A COURSE OF
GEOMETRY

A COURSE OF GEOMETRY

BY

R. N. SEN, M.A., PH.D., F.N.I.

PROFESSOR OF PURE MATHEMATICS,
CALCUTTA UNIVERSITY

CALCUTTA CENTRAL LIBRARY UNIVERSITY

PUBLISHED BY THE
CALCUTTA UNIVERSITY
1960

First Edition 1953.
Second Edition 1960.

573
De 5·5
c3

222859

TSCU 2691

PRINTED IN INDIA

PRINTED AND PUBLISHED BY SIBENDRANATH KANJILAL,
SUPERINTENDENT, CALCUTTA UNIVERSITY PRESS,
48, HAZRA ROAD, BALLYGUNGE, CALCUTTA.

1956B—August, 1960—A

Price Rs. 15/-

PREFACE TO THE FIRST EDITION

The book is an outcome of a course of lectures delivered by me about fifteen years ago to the students of the department of Pure Mathematics, Calcutta University, in accordance with the changed syllabus which was introduced then. In making the changes in the course of study, previous training of the students had naturally to be taken into account, and lectures were prepared in collaboration with Professor F. W. Levi who was then the head of the department. As a matter of fact, notes on lectures, taken down by one of his students, which Prof. Levi had delivered previously at the University of Leipzig were in my hands and I made use of them. Among the published works on the subject which I have consulted, Graustein's "Introduction to Higher Geometry" is to be specially mentioned.

The book is built around Klein's classification of geometries and is divided into two parts, the plane and the space. In the former, development proceeds from the metric to the projective while in the latter, a somewhat opposite course is taken just to indicate that the subject can be developed either way. Although synthetic method has been used occasionally for fixing up certain concepts in mind, the book is mainly an analytic geometry. As such it requires certain indispensable algebraic tools, and these have been provided for at the outset under Basic Algebra. Here again, the first chapter of Levi's Algebra, Vol. I, has been freely consulted. The defects and limitations of the book are certainly not due to him, but whatever success it may attain would be due to his efforts in initiating a coordinated scheme.

I wish to express my best thanks to my colleagues Mr. B. C. Chatterjee and Mr. M. C. Chaki who have supplied the examples given at the end of the book and prepared the index. I wish also to record my appreciation to the Superintendent Mr. S. N. Kanjilal and his staff of the Calcutta University Press for the care and pains they have taken in the making of the book.

R. N. Sen

PREFACE TO THE SECOND EDITION

The first edition of the book was exhausted within a few years and the book remained out of print for quite sometime. As the demand for the book was growing in the meantime, a second edition had to be undertaken with some little changes here and there.

I express my hearty thanks to Dr. M. C. Chaki who went through the proofs. I also thank the Superintendent and the staff of the Calcutta University Press for their care and efforts in bringing out this edition.

Dept. of Pure Mathematics.
Calcutta University

R. N. Sen

CONTENTS

Page

CHAPTER 0. BASIC ALGEBRA

1. Vector space . . . xiii
2. Matrices . . . xv
3. Systems of linear equations . . . xviii
4. Determinants . . . xxii
5. Products of matrices and of determinants . . . xxix
6. Linear transformations . . . xxxii
7. Groups . . . xxxv

PLANE GEOMETRY

CHAPTER I. VECTORS AND ANGLES

1. Points and vectors on a straight line . . . 1
2. Points and vectors in the plane . . . 3
3. Angle . . . 6
4. The straight line . . . 10
4.1. Hessian normal form and the perpendicular distance . . . 11
5. Straight line and triangle . . . 12

CHAPTER II. CROSS-RATIO

6. Cross-ratio of four collinear points . . . 15
7. Cross-ratio of four concurrent lines . . . 16
7.1. Cross-ratio of projection and section . . . 18
8. The six cross-ratios of the twenty-four permutations . . . 20
8.1. Special cases . . . 21
9. Harmonic division . . . 22

CHAPTER III. RIGID MOTIONS

10. Change of coordinate axes . . . 24
10.1. Invariants of the transformation . . . 25
11. Translation and rotation . . . 26
11.1. Product of rigid motions . . . 27

Page

CHAPTER IV. **CONICS**

12. Classification of conics and their equations in normal forms 29
13. Pole and polar. Tangent 35
14. Focus and directrix 38
14.1. The complex plane 42

CHAPTER V. **TRANSFORMATIONS OF SYMMETRY AND SIMILARITY**

15. Symmetry 47
15.1. Existence of fixed points 50
16. Similarity 51

CHAPTER VI. **THE CIRCLE**

17. Power of a point with respect to a circle 54
18. Pole, polar 56
19. Coaxal system 56
19.1. Types of coaxal system. Orthogonal system 58
20. Centres of similitude of two circles 60
21. Inversion 62
22. Polar reciprocation with respect to a circle 65

CHAPTER VII. **AFFINITY**

23. Affine transformations 70
23.1. Particular cases 73
23.2. Some invariants 75
24. Affine classification and properties of nondegenerate conics 76
24.1. Conjugate diameters 78
24.2. Transformation of conjugate diameters by simple strain . 83

CHAPTER VIII. **INVOLUTION**

25. Projective pencils of lines 86
26. Involution of lines 88
27. Projective rows. Involution of points 91
27.1. Involution of conjugate points and lines with respect to a nondegenerate conic 94

CHAPTER IX. **GEOMETRY IN THE EXTENDED CARTESIAN PLANE**

28. Points and line at infinity 96
29. Projective pencils and rows 97
29.1. Complete quadrangle and complete quadrilateral . . . 101
30. Homogeneous coordinates of a point 104

Page

30.1. Linear dependence of points and lines 105
31. Homogeneous line coordinates 108
32. Principle of duality 111
33. Loci and envelopes 113

CHAPTER X COLLINEATION AND CORRELATION

34. Transformations of collineation 115
34.1. Properties of collineations 117
35. One-dimensional linear transformations 119
36. Generalisation by collineation 121
37. Correlation and polarity 127

CHAPTER XI. GEOMETRY IN THE PROJECTIVE PLANE

38. Projective coordinates 131
38.1. Transformation of projective coordinates. Collineation . 134
39. Classification of polarities and conics in the projective plane 136
40. Quadratic dependence of points 141
41. Projective theory of conics 146
42. Pencil of conics 156
42.1. Invariants of two conics 166
43. Affine transformations in projective geometry 168
44. Metric properties in the projective plane 171

APPENDIX I. TRILINEAR AND AREAL COORDINATES 175

SPACE GEOMETRY

CHAPTER XII. THE EUCLIDEAN SPACE

45. Points and vectors 179
46. Scalar product of two vectors. Area of a triangle . . . 181
47. The straight line 183
48. The plane 185
48.1. Vector product of two vectors 188
49. Intersection of planes 189
50. The tetrahedron 191
51. Projection 193

CHAPTER XIII. PROJECTIVE SPACE

52. Principal notions 196
52.1. Projective plane and plane projective geometry 201

Page

53. Projective space as an extension of Euclidean space . . 202
54. Pluecker coordinates of a line 203
55. Projectivity. Cross-ratio 206
55.1. Homology 211

CHAPTER XIV. GROUPS OF TRANSFORMATIONS AND CLASSIFICATION OF GEOMETRIES

56. Dimensionality of spaces 214
57. Groups of transformations 216
58. Classification of geometries 221

CHAPTER XV. PROJECTIVE THEORY OF QUADRICS

59. Projective properties of quadrics 224
60. Collineation 226
61. Projective classification of quadrics 229
62. Projective generation of ruled quadrics 233

CHAPTER XVI. POLARITY

63. Correlation 236
64. Polarity and null system 237
65. Transformation of correlation by collineation 239
66. Polar fields 242

CHAPTER XVII. GEOMETRY IN THE EXTENDED CARTESIAN SPACE

67. The Circle at infinity 245
68. Orthogonality as polarity 247
69. Affine properties of quadrics 249

CHAPTER XVIII. ORTHOGONAL TRANSFORMATION AND AFFINITY

70. Change of coordinate axes 251
71. Rigid motion and symmetry 252
71.1. Geometrical properties of orthogonal transformations . . 256
72. Affine transformations 257

CHAPTER XIX. QUADRICS IN EUCLIDEAN SPACE

73. Pole, polar. Tangent 262
74. Transformation of the general equation 263
75. Metric classification of quadrics 265
76. Affine classification of quadrics 273

Page

77. Generators of the hyperboloid of one sheet and of the hyperbolic paraboloid 274
78. Plane sections of quadrics 277
79. Circular sections 279
80. Confocal quadrics 284
81. Surfaces of revolution and ruled surfaces 286

CHAPTER XX. **LAW OF INERTIA FOR QUADRATIC FORMS**

82. Homogeneous quadratic forms 290
83. Law of inertia 292
84. Geometrical significance 293
EXAMPLES 295
INDEX 308

CHAPTER 0

BASIC ALGEBRA

Those topics of algebra, a knowledge of which has been assumed in the analytical method generally used in the book, are the theories of determinants and systems of linear equations. Although the former had its origin in the latter, both of them are now usefully made to depend on the theory of matrices which, in turn, is intimately connected with the theory of vectors. It is intended to give here a brief account of those features of these and other theories which constitute what may be called the 'basic algebra' of this book.

1. Vector space. An ordered set of n numbers, where n is a positive integer, is called an *n-vector* and is written

$$\alpha = (a_1, \ldots, a_n) \tag{1}$$

The n-vector α is uniquely defined by the ordered numbers $a_1, \ldots, a_n$, not necessarily all distinct, which are called its successive *coordinates*.

If c is any number, the *product* of c and α is defined as the n-vector

$$c\alpha = (ca_1, \ldots, ca_n)$$

Further, if $\beta = (b_1, \ldots, b_n)$ is an n-vector, the *sum* of α and β is defined as the n-vector

$$\alpha + \beta = (a_1 + b_1, \ldots, a_n + b_n)$$

Obviously $\alpha = \beta$ if and only if $a_1 = b_1, \ldots, a_n = b_n$ and then $\alpha + \beta = 2\alpha$.

We have the following special n-vectors :

$$\begin{gathered} 0 = (0, \ldots, 0), \\ \epsilon_1 = (1, 0, \ldots, 0), \quad \epsilon_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \epsilon_n = (0, \ldots, 0, 1) \end{gathered} \tag{2}$$

They are called the *zero vector* and the successive *n unit vectors*.

Now let $\alpha_1, \ldots, \alpha_m$ be given m n-vectors, m being a positive integer. Form with them a linear combination

$$\sum_{i=1}^{m} c_i\alpha_i, \qquad \text{where} \quad c_1, \ldots, c_m \quad \text{are numbers.}$$

It follows from the above definitions that this linear combination is an n-vector. The vectors $\alpha_1, \ldots, \alpha_m$ are then said to be a set of *independent* vectors if $\sum c_i\alpha_i = 0$ is satisfied by no values of $c_1, \ldots, c_m$ other than all zero. And $\alpha_1, \ldots, \alpha_m$ are said to be a set of *dependent* vectors if there exist $c_1, \ldots, c_m$, not all zero, such that $\Sigma\, c_i\alpha_i = 0$ holds. Suppose $\alpha_1, \ldots, \alpha_m$ are dependent vectors and $c_1 \neq 0$. Then we can express α_1 as $\alpha_1 = \sum_{j=2}^{m} d_j\, \alpha_j$. In this case α_1 is said to *depend on* $\alpha_2, \ldots, \alpha_m$. In general, if $\beta = \Sigma\, c_i\alpha_i$, then β depends on the given α's whatever values $c_1, \ldots, c_m$ may have. Different sets of values of the c's give generally different β's.

It follows from these definitions that a single vector, other than 0, is independent and the vector 0 is always a dependent vector, depending on any set of vectors.

The set of all n-vectors which depend on a given set of n-vectors is said to form a *vector space* generated by the given set of vectors. A set of independent n-vectors which generate a vector space V is called a *basis* of V and the number of vectors in a basis is called the *rank* of V. The vector space generated by 0 (and therefore consisting of 0 only) has no basis, but it may be said to be of rank zero. Finally, let V and V' be two vector spaces; if every vector of V' belongs to V, then V' is called a *subspace* of V.

THEOREM 1. *Every vector space, whose rank is not zero, has a basis. A basis of a vector space is not unique but its rank is.*

Proof: Let a vector space V be generated by $\alpha_1, \ldots, \alpha_m$, not all 0. If these generating vectors do not form a basis, at least one of them, say α_m, is dependent on the others. So V can be generated by $\alpha_1, \ldots, \alpha_{m-1}$. Proceeding in this way we can, after a finite number of steps, so reduce the number of generating vectors that the remaining vectors just form an independent set, *i.e.*, a basis of V.

Again, let $\alpha_1, \ldots, \alpha_r$ be a basis of V and $\beta = \sum_{1}^{r} c_i\alpha_i$, where $c_r \neq 0$. Then $\alpha_1, \ldots, \alpha_{r-1}, \beta$ are independent vectors of V. For, let

$$d_1\alpha_1 + \ldots + d_{r-1}\alpha_{r-1} + d_r\beta = 0$$

Then $$(d_1 + d_rc_1)\alpha_1 + \ldots + (d_{r-1} + d_rc_{r-1})\,\alpha_{r-1} + d_rc_r\alpha_r = 0$$

Since the α's are independent, so $d_1 = \ldots = d_r = 0$ and hence $\alpha_1, \ldots, \alpha_{r-1}, \beta$ are independent. Now let V' be the vector space generated by $\alpha_1, \ldots, \alpha_{r-1}, \beta$. Then V' is a subspace of V. On the other hand, V is a subspace of V'. Therefore $V = V'$. Hence $\alpha_1, \ldots, \alpha_{r-1}, \beta$ is a basis of V which shows that V can have more than one basis.

Finally, let $\beta_1, \ldots, \beta_t$ be a set of independent vectors belonging to V, V' be the vector space generated by them and, as before, let $\alpha_1, \ldots, \alpha_r$ be a basis of V. If possible, suppose $t > r$. We can, as before, replace one of the α's, say α_1, by β_1 and get $\beta_1, \alpha_2, \ldots, \alpha_r$ as a basis of V. As β_2 belongs to V, it must be possible to express β_2 as

$$\beta_2 = c_1\beta_1 + c_2\alpha_2 + \ldots + c_r\alpha_r$$

And as β_1, β_2 are independent, at least one of the numbers $c_2, \ldots, c_r$ is different from zero, say $c_2 \neq 0$. Therefore $\beta_1, \beta_2, \alpha_3, \ldots, \alpha_r$ form another basis of V. Proceeding in this way we can have, after a finite number of steps, a new basis of V consisting of r β's, say $\beta_1, \ldots, \beta_r$. It follows that V is a subspace of V' whereas V' is not a subspace of V. This means that $\beta_1, \ldots, \beta_t$ cannot all belong to V which is contrary to hypothesis. Hence $t \ngtr r$. This shows that the maximum number of independent vectors of V is r. Accordingly, the rank of V is constant and is equal to r.

THEOREM 2. *Let V' be a subspace of a vector space V of n-vectors and let their ranks be r' and r respectively. The $r' \leq r \leq n$.*

Proof: From the proof of the last theorem it is easily seen that a basis of V can be obtained which will include a basis of V'. Therefore $r' \leq r$.

Again, let the n unit n-vectors $\epsilon_1, \ldots, \epsilon_n$ defined by (2) generate the vector space W. As these vectors are independent, the rank of W is n. Moreover, W contains every n-vector. For, if $\alpha = (a_1, \ldots, a_n)$ is any n-vector, α can be expressed as $\alpha = \Sigma\, a_i \epsilon_i$. Therefore V is a subspace of W. Hence $r \leq n$.

THEOREM 3. *Let Γ be a set of n-vectors with the properties that (i) the sum of two vectors of Γ belongs to Γ and (ii) the product of a number and a vector of Γ belongs to Γ. Then Γ is a vector space.*

Proof: Let the maximum number of independent vectors of Γ be r. If $\alpha_1, \ldots, \alpha_r$ is a set of independent vectors of Γ, then, by virtue of the properties (*i*) and (*ii*), the vector $\sum_1^r c_i \alpha_i$ belongs to Γ. Hence Γ is the vector space generated by $\alpha_1, \ldots, \alpha_r$.

2. Matrices. A rectangular scheme or array composed of mn numbers which are arranged in m rows and n columns is called an $m \times n$ *matrix*, and is generally written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & , & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & & \cdot & \cdot & a_{mn} \end{pmatrix} \tag{3}$$

The numbers a_{ij} ($i = 1, \ldots, m$; $j = 1, \ldots, n$), which are not necessarily all distinct, are called the *constituents* of the matrix A. Any matrix is uniquely defined by its ordered constituents arranged in rows and columns. If every constituent of a matrix is zero, it is called a *zero matrix* and if $m = n$, it is called a *square matrix*.

The successive rows of A can be regarded as an ordered set of m n-vectors and its successive columns as an ordered set of n m-vectors. Let the row-vectors of A be denoted by $\alpha_1, \ldots, \alpha_m$ and its column-vectors by $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$. So

$$\alpha_i = (a_{i1}, \ldots, a_{in}), \qquad \bar{\alpha}_k = (a_{1k}, \ldots, a_{mk}) \qquad \begin{array}{l} i = 1, \ldots, m \\ k = 1, \ldots, n \end{array}$$

Also, let the vector space generated by the row-vectors of the matrix A be denoted by $R(A)$ and that generated by its column-vectors be denoted by $C(A)$.

Consider the three types of operations denoted by $\varphi_1, \varphi_2, \varphi_3$ (say), which can be applied on one or more of the vectors $\alpha_1, \ldots, \alpha_m$ and called row-multiplication, row-addition, row-omission. They are defined as follows :

$$\begin{array}{ll} \varphi_1 & \text{is an operation of multiplication of } \alpha_i \text{ by a number } b \neq 0 \\ \varphi_2 & \text{is an operation of addition to } \alpha_i \text{ of } c\alpha_j \\ \varphi_3 & \text{is an operation of omission of } \alpha_i \text{ if } \alpha_i = 0 \end{array} \tag{4}$$

These operations (4) constitute a class of transformations, applied on the row-vectors of a matrix, called *sweep-out* transformations.

THEOREM 4. *The ranks of $R(A)$ and $C(A)$ remain unaltered by the application of a sweep-out transformation.*

Proof : From what has been said in §1 it follows that the vector space generated by two vectors, say α and β, is the same as that generated by $b\alpha$ $b \neq 0$) and β or that generated by $\alpha + c\beta$ and β or by only β if $\alpha = 0$. Therefore $R(A)$ remains unchanged by a sweep-out transformation and hence its rank remains unaltered.

Again, as $C(A)$ is generated by $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$, any vector λ of $C(A)$ is given by

$$\lambda = \sum_{t=1}^{n} d_t \bar{\alpha}_t = (\sum_t d_t a_{1t}, \ldots, \sum_t d_t a_{mt})$$

Therefore the ith coordinate of λ is changed by the applications of the operations φ_1 and φ_2 into $b \sum d_t a_{it}$ and $\sum d_t (a_{it} + ca_{jt})$ respectively and by φ_3 the ith coordinate is altogether omitted if α_i is 0. Suppose then that λ is transformed into λ' by any sweep-out transformation (4). It follows that if, for a given set of values of $d_1, \ldots, d_n$, the vector $\lambda = 0$ (or $\neq 0$), then also $\lambda' = 0$ (or $\neq 0$) for the same set of the d's. This shows that if the column-vectors were originally independent (dependent), then, after the application of a sweep-out transformation, their transforms would also remain independent (dependent). Now suppose that the rank of $C(A)$ is r and, without loss of generality, suppose that the first r column-vectors are independent. Then, after the application of a sweep-out transformation, the first r of the transformed vectors would remain independent and the others would depend on them. Hence the rank of $C(A)$ remains unaltered.

It is to be noticed that not only the rank of $R(A)$ but $R(A)$ itself remains unaltered. This is however not true of $C(A)$, as the m-vectors α_k may be changed into p-vectors, where $p < m$, in view of the operation φ_3.

THEOREM 5. *By applications of sweep-out transformations any $m \times n$ matrix, other than a zero matrix, can be transformed into the matrix*

$$E = \begin{pmatrix} 1\,0\ldots0 & e_{11}\ldots e_{1\,n-r} \\ 0\,1\,0\ldots0 & e_{21}\ldots e_{2\,n-r} \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ 0\,0\ldots0\,1 & e_{r1}\ldots e_{r\,n-r} \end{pmatrix} \tag{5}$$

or into a matrix which can be obtained from E by permutation of columns, where E is an $r \times n$ matrix, $r \leq m$, and e_{ij} are numbers which would depend on the constituents of the given matrix.

Proof: Take the matrix A given by (3) whose row-vectors are, as before, denoted by $\alpha_1, \ldots, \alpha_m$. In view of the operation φ_3, we may, without loss of generality, suppose that none of these row-vectors is 0. Let a_{1s} be a coordinate of α_1 which is not 0. Replace α_1 by α_1/a_{1s} by the application of the operation φ_1. Then by operation φ_2, replace α_2 by $\alpha_2 - (a_{2s}/a_{1s})\,\alpha_1$, σ_3 by $\alpha_3 - (a_{3s}/a_{1s})\alpha_1, \ldots, \alpha_m$ by $\alpha_m - (a_{ms}/a_{1s})\alpha_1$. If by these operations any row-vector of A is transformed into 0, strike it out by application of φ_3. Denote the matrix into which A is now transformed by A' and its row-vectors by $\alpha'_j = (a'_{j1}, \ldots, a'_{jn}), j = 1, 2, \ldots$ It is easily seen that the sth column-vector of A' is equal to the 1st column-vector of E. As before, let a'_{2t} be a coordinate of α'_2 which is not 0. Replace α'_2 by α'_2/a'_{2t} and then replace α'_1 by $\alpha'_1 - (a'_{1t}/a'_{2t})\alpha'_2$, α'_3 by $\alpha'_3 - (a'_{3t}/a'_{2t})\alpha'_2, \ldots$. If by these operations any row-vector of A' is transformed into 0, strike it out by the application of φ_2. Denote the matrix into which A' is now transformed by A''. It is easily seen that the sth and tth column-vectors of A'' are equal to the 1st and 2nd column-vectors of E. Proceeding in this manner we can, after a finite number of steps, transform the matrix A into a matrix of the form stated in the theorem.

Denote the desired matrix of the theorem into which A is transformed (*i.e.*, either E or one obtained from it by permutation of columns only) by S. Then A is said to be *swept-out* into S.

THEOREM 6. *Rank of $R(A)$ = rank of $C(A)$, for every matrix A.*

Proof: We first show that the rank of $R(E)$ = the rank of $C(E)$ = the number of rows of E which is supposed to be r. Let the row-vectors of E be denoted by $\beta_1, \dots, \beta_r$. Then any vector ϵ of $R(E)$ can be expressed as

$$\epsilon = \sum_{i=1}^{r} c_i\beta_i = (c_1, \dots, c_r, c_1 e_{11} + \dots + c_r e_{r1}, \dots, c_1 e_{1\,n-r} + \dots + c_r e_{r\,n-r})$$

Therefore $\epsilon = 0$ if and only if $c_1 = \dots = c_r = 0$. Hence the row-vectors of E are independent and accordingly the rank of $R(E) = r$. Again, it can be easily seen that the first r column-vectors of E are independent and the others depend on them. Therefore the rank of $C(E) = r$.

Now if A is a zero matrix, both $R(A)$ and $C(A)$ are of rank 0, If not, let A be swept-out into S. By theorem 4, the ranks of $R(A)$ and $C(A)$ are not thereby altered. As any permutation of the columns of E does not alter the rank of $C(E)$, the rank of $C(E)$ = the rank of $C(S)$. Therefore the rank of $R(A)$ = the rank of $C(A) = r$.

Definition. The rank of $R(A)$ (or of $C(A)$) of a matrix A is called the *rank of the matrix A*.

3. Systems of linear equations. (I) *Homogeneous systems.*

Let an arbitrarily given linear and homogeneous system of equations in the unknowns $x_1, \dots, x_n$ be

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \qquad & \qquad (6) \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

The known numbers a_{ij} are called the *coefficients* of the equations of the system and the matrix A defined by (3) of these coefficients is the matrix of the system. A set of values $x_1 = \alpha_1, \dots, x_n = \alpha_n$ of the x's which satisfy every equation of the system (6) is a *solution* of the system. A solution may therefore be regarded as an n-vector $\alpha = (a_1, \ldots, a_n)$. In particular, 0 is a solution of every homogeneous system.

Theorem 7. *The solutions of a homogeneous system form a vector space.*

Proof : Let the homogeneous system be given by (6) and let α and β be two solutions of the system. Then obviously $c\alpha$ and $\alpha + \beta$ are also solutions of the system. Therefore, by theorem 3, the set of all solutions form a vector space.

Denote the vector space of the solutions of the system (6) by $X(A)$.

Two systems of equations are said to be *equivalent* when every solution of either system is a solution of the other.

Theorem 8. *If the matrix A is swept-out into the matrix S, then the system (6) is equivalent to the system whose coefficients form the matrix S.*

Proof : Let the successive equations of the system (6) be denoted by $f_1 = 0, f_2 = 0, \ldots, f_m = 0$ and let α be a solution of the system. Then α is also a solution of the equations $bf_i = 0$ $(b \neq 0)$ and $f_i + df_j = 0$. Further, the trivial equation whose coefficients are all zero can always be removed from any system without affecting the solutions of the system. Therefore the solutions of the system (6) will not be affected if $f_i = 0$ is replaced by $bf_i = 0$ or by $f_i + df_j = 0$ or if $f_i = 0$ is altogether omitted when its coefficients are all zero. Now the operations of replacing f_i by $bf_i, f_i + df_j$ and of removing f_i if its coefficients are all zero are operations of sweep-out transformations by which A is transformed into S. It therefore follows that every solution of the system (6) is a solution of the system whose coefficients form the matrix S. Conversely, every solution of the latter system is a solution of (6). For, it is possible to retrace the steps by which A is transformed into S and thereby get back the system (6) without affecting the solutions.

Theorem 9. *The rank of $R(A)$ + the rank of $X(A) = n$.*

Proof : Let the rank of $R(A)$ be r. Then the matrix S into which A is swept-out has r rows. Since we are concerned with ranks only, we may, without loss of generality and for the sake of definiteness, suppose that $S = E$ as given by (5). Then, by the last theorem, the system (6) is equivalent to the system

$$\begin{aligned}
x_1 + e_{11} x_{r+1} + \ldots + e_{1\,n-r} x_n &= 0 \\
x_2 + e_{21} x_{r+1} + \ldots + e_{2\,n-r} x_n &= 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\
x_r + e_{r1} x_{r+1} + \ldots + e_{r\,n-r} x_n &= 0
\end{aligned}$$

Now, for solution of this system, we may choose the values of $x_{r+1}, \ldots, x_n$ arbitrarily and when they are once chosen, the values of $x_1, \ldots, x_r$ are uniquely determined by them. Therefore, for each set of values of $(x_{r+1}, \ldots, x_n)$ arbitrarily chosen, we obtain a solution $(x_1, \ldots, x_r, x_{r+1}, \ldots, x_n)$. Further $(x_{r+1}, \ldots, x_n)$ is an $(n-r)$-vector and, by theorem 2, the maximum number of such vectors which are independent is $n-r$. Therefore the maximum number of independent solutions is $n-r$, *i.e.*, the rank of $X(A) = n-r$. Hence, the rank of $R(A)$ + the rank of $X(A) = n$.

In practice, the $n-r$ independent $(n-r)$-vectors $(x_{r+1}, \ldots, x_n)$ and the corresponding $n-r$ independent solutions, say $\beta_1, \ldots, \beta_{n-r}$, of the system (6) are chosen as follows :

Independent $(n-r)$-vectors $(x_{r+1}, \ldots, x_n)$	Independent solutions $(x_1, \ldots, x_r, x_{r+1}, \ldots, x_n)$
$(1, 0, \ldots \ldots, 0)$	$(-e_{11}, \ldots, -e_{r1}, 1, 0, \ldots, 0) = \beta_1$
$(0, 1, \ldots \ldots, 0)$	$(-e_{12}, \ldots, -e_{r2}, 0, 1, \ldots, 0) = \beta_2$
. . . .	. . . .
$(0, 0, \ldots \ldots, 1)$	$(-e_{1\,n-r}, \ldots, -e_{r\,n-r}, 0, 0, \ldots, 1) = \beta_{n-r}$

All solutions of the given system (6) are then given by

$c_1 \beta_1 + \ldots + c_{n-r} \beta_{n-r}$, where $c_1, \ldots, c_{n-r}$ are arbitrary numbers.

It may be seen that 0 is included in these solutions.

(II) *Nonhomogeneous systems.* Let an arbitrarily given linear and nonhomogeneous system of equations in the unknowns $x_1, \ldots, x_n$ be

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + a_{10} &= 0 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n + a_{20} &= 0 \\ \ldots \ldots \ldots \ldots \ldots & \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n + a_{m0} &= 0 \end{aligned} \tag{7}$$

where the numbers $a_{10}, \ldots, a_{m0}$ are not all zero. Along with the above system consider the homogeneous system (6) with the corresponding matrix A given by (3). Also, introducing a new unknown quantity x_0, consider the following homogeneous system in $n+1$ unknowns together with the corresponding matrix A_0

$$\begin{array}{l} a_{11}x_1 + \dots\dots + a_{1n}x_n + a_{10}x_0 = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{m1}x_1 + \dots\dots + a_{mn}x_n + a_{m0}x_0 = 0 \end{array} \qquad A_0 = \begin{pmatrix} a_{11} \dots\dots a_{1m}\, a_{10} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{m1} \dots a_{mn}\, a_{m0} \end{pmatrix} \qquad (8)$$

As before, a solution of the system (7) is an n-vector $(x_1, \dots, x_n)$ and a solution of the system (8) is an $(n+1)$-vector $(x_1, \dots, x_n\ x_0)$. As the systems (6) and (8) are homogeneous systems, so, by theorem 7, their solutions form vector spaces denoted respectively by $X(A)$ and $X(A_0)$. Also, as before, the rank of $R(A_0)$ = the rank of $C(A_0)$ = the rank of A_0.

THEOREM 10. *The system* (7) *has solution if and only if the rank of* A = *the rank of* A_0.

Proof: Consider the vectors of $X(A_0)$. Let us divide these $(n+1)$-vectors into two classes. Those vectors whose last coordinates are zero are taken to form class 1, and the remaining vectors form class 2. The vectors of class 2 are independent of the vectors of class 1, because no vector of class 2 can be made to depend on any set of vectors of class 1. Now class 1 exists. For, if $(a_1, \dots, a_n)$ is a vector of $X(A)$, then $(a_1, \dots, a_n, 0)$ is a vector of class 1; and as $X(A)$ exists, so class 1 exists. Moreover, because of this existence of a one-to-one correspondence between the vectors of $X(A)$ and those of class 1, namely $(a_1, \dots, a_n) \longleftrightarrow (a_1, \dots, a_n, 0)$, the vectors of class 1 form a vector space whose rank is equal to the rank of $X(A)$.

Further, if class 2 exists, then a solution of the system (7) exists. For, if $(b_1, \dots, b_n, b_0)$, $b_0 \neq 0$, is a vector of class 2, then $(b_1/b_0, \dots, b_n/b_0)$ is a solution of the system (7). Conversely, if a solution of the system (7) exists, class 2 exists. For, if $(c_1, \dots, c_n)$ is a solution of the system (7), then $(c_1, \dots, c_n, 1)$ is a vector of class 2.

Now let the rank of A be r. Then the rank of A_0 is either r or $r+1$, because A_0 is obtained from A by introducing one more (the last) column and this column may or may not be dependent on the other columns. Therefore, by theorem 9, the rank of $X(A_0)$ is either $n+1-r$ or $n-r$. But the vectors of the classes 1 and 2 taken together constitute $X(A_0)$ of which class 1 is of rank $n-r$ and is independent of class 2. Therefore, as the system (7) has solution if and only if class 2 exists, it follows that the system (7) has solution if and only if the rank of $X(A_0)$ is $n+1-r$, *i.e.*, if and only if the rank of $A_0 = r =$ the rank of A.

Theorem 11. *If α is a solution of system (7), then all solutions of system (7) are obtained by adding to α all solutions of system (6).*

Proof : If α' is an arbitrary solution of (7), then $\alpha'-\alpha=\beta$ is a solution of (6). It follows that if β is an arbitrary solution of (6), $\alpha'=\alpha+\beta$ is a solution of (7).

In practice, to find all solutions of (7) when it has a solution, we proceed as follows :

We first find all solutions of (6) as in (I) above. Let these solutions be those given there, namely the n-vectors

$$c_1\beta_1+\;\ldots\;+c_{n+r}\,\beta_{n+r}$$

We then look for a solution of (7). In order to do so, we sweep-out the matrix A_0 and obtain the system which is equivalent to (8) as we have done in (I). Let this system be

$$x_1+e_{11}\,x_{r+1}+\;\ldots\;+e_{1n-r}\,x_n+e_{1n-r+1}\,x_0=0$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$x_r+e_{r1}\,x_{r+1}+\;\ldots\;+e_{rn-r}\,x_n+e_{rn-r+1}\,x_0=0$$

Then the $(n+1)$-vector $\xi=(-e_{1n+r+1},\ldots,-e_{rn+r+1},0,\ldots,0,1)$ is a solution of (8) in which the last coordinate is not zero. It follows that the n-vector

$$\alpha=(-e_{1n-r+1},\ldots,-e_{rn-r+1},0,\ldots,0)$$

is a solution of (7). Therefore all solutions of system (7) are given by

$$\alpha+c_1\,\beta_1+\ldots+c_{n+r}\,\beta_{n+r}$$

Corollary. Let $m=n$; then *the system (6) has the only solution 0 and the system (7) has exactly one solution if and only if the rank of A is n.*

Proof : The first part follows from theorem 9. Regarding the second part, it follows from theorems 10, 11 that if the rank of A is n, the system (7) has exactly one solution and if the system (7) has exactly one solution, the rank of A_0 = the rank of $A=n$.

4. Determinants. Take a square matrix of n rows and columns

$$A=\begin{pmatrix}a_{11}\ldots\ldots a_{1n}\\ \cdot\quad\cdot\quad\cdot\\ a_{n1}\ldots\ldots a_{nn}\end{pmatrix}=(a_{ij}) \tag{9}$$

As before, let $\alpha_1,\ldots,\alpha_n$ denote the successive row-vectors of A. Any matrix is a special kind of arrangement of some numbers but is itself not a number. We shall now consider a *function* of the square matrix A, *i.e.*, a function of the n^2 numbers a_{ij} or of the n n-vectors $\alpha_1, \ldots, \alpha_n$, which shall be a number. This particular function which is defined below is called the *determinant* and is denoted by any one of the following notations :

$$\det A = \det(a_{ij}) = \det(\alpha_1, \ldots, \alpha_n) = \begin{vmatrix} a_{11} \ldots a_{1n} \\ \cdot \quad \cdot \quad \cdot \\ a_{n1} \ldots a_{nn} \end{vmatrix} = | a_{ij} | \tag{10}$$

The determinant of A is defined by the following three properties :

(*a*) If α_i is multiplied by a number c, det A is multiplied by c, *i.e.*,

$$\det(\alpha_1, \ldots, c\alpha_i, \ldots, \alpha_n) = c \det(\alpha_1, \ldots, \alpha_n), \; 1 \leq i \leq n$$

(*b*) If α_i is replaced by $\alpha_i + \alpha_j$, $i \neq j$, det A remains unaltered, *i.e.*,

$$\det(\alpha_1, \ldots, \alpha_i + \alpha_j, \ldots, \alpha_n) = \det(\alpha_1, \ldots, \alpha_n), \quad 1 \leq i \neq j \leq n$$

(*c*) If $\alpha_1 = \epsilon_1, \ldots, \alpha_n = \epsilon_n$, where the ϵ_i's are defined by (2), det A has the value unity, *i.e.*,

$$\det(\epsilon_1, \ldots, \epsilon_n) = 1$$

THEOREM 12. *If α_i is replaced by $\alpha_i + c\alpha_j$, $i \neq j$, det A is not altered. If α_i and α_j are interchanged, $i \neq j$, det A is changed into its negative.*

Proof : The first part follows directly from properties (*a*), (*b*). For, if $c \neq 0$ and $i < j$

$$\det A = \frac{1}{c} \det(\alpha_1, \ldots, \alpha_i, \ldots, c\alpha_j, \ldots, \alpha_n)$$

$$= \frac{1}{c} \det(\alpha_1, \ldots, \alpha_i + c\alpha_j, \ldots, c\alpha_j, \ldots, \alpha_n) = \det(\alpha_1, \ldots, \alpha_i + c\alpha_j, \ldots, \alpha_n)$$

Regarding the second part, suppose, without loss of generality, that $i = 1$. Then

$$\begin{aligned} \det A &= \det(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_n) = \det(\alpha_1, \ldots, \alpha_j + \alpha_1, \ldots, \alpha_n), \text{ by } (b) \\ &= \det(\alpha_1 - \alpha_j - \alpha_1, \ldots, \alpha_j + \alpha_1, \ldots, \alpha_n), \text{ by the first part} \\ &= \det(-\alpha_j, \ldots, \alpha_j + \alpha_1, \ldots, \alpha_n) \\ &= \det(-\alpha_j, \ldots, \alpha_j + \alpha_1 - \alpha_j, \ldots, \alpha_n), \text{ by } (b) \\ &= \det(-\alpha_j, \ldots, \alpha_1, \ldots, \alpha_n) = -\det(\alpha_j, \ldots, \alpha_1, \ldots, \alpha_n), \text{ by } (a). \end{aligned}$$

COROLLARY. *If $\alpha_i = 0$ or if $\alpha_i = \alpha_j$, $i \neq j$, then det $A = 0$*

Proof: The first part follows from property (*a*), because $0 = 0\alpha$, where α is an arbitrary vector. And the second part follows from the fact that as det $A = -$det A, so det $A = 0$.

THEOREM 13. *If* $\alpha_1, .., \alpha_n$ *is a set of dependent vectors, det* $A = 0$.

Proof: If $\alpha_1, .., \alpha_n$ are dependent, we must have $\Sigma\, c_k\alpha_k = 0$, where the c's not all zero. So, without loss of generality, let $c_1 \neq 0$. Then $\alpha_1 = \sum_{j=2}^{n} d_j\, \alpha_j$. Therefore, by the last theorem,

$$\det A = \det(\alpha_1, .., \alpha_n) = \det(\alpha_1 - d_2\alpha_2, \alpha_2, .., \alpha_n)$$

$$= \det(\alpha_1 - d_2\alpha_2 - d_3\alpha_3, \alpha_2, .., \alpha_n) = \ldots = \det(\alpha_1 - \Sigma\, d_j\alpha_j, \alpha_2, .., \alpha_n)$$

$$= \det(0, \alpha_2, .., \alpha_n) = 0$$

THEOREM 14. *Let* $B_1, .., B_t$ *be the matrix obtained from A by replacing* α_i *by the n-vectors,* $\beta_1, .., \beta_t$ *respectively and let* $\alpha_i = c_1\beta_1 + \,..\, c_t\beta_t$. *Then*

$$det\ A = c_1\, det\ B_1 + \,..\, + c_t\, det\ B_t$$

Proof: Let us first consider the particular case when $\alpha_i = \beta_1 + \beta_2$. Suppose, without loss of generality, that $i = 1$. So we have to show

$$\det(\beta_1 + \beta_2, \alpha_2, .., \alpha_n) = \det(\beta_1, \alpha_2, .. \alpha_n) + \det(\beta_2, \alpha_2, .., \alpha_n)$$

Two cases arise : (1) $\beta_1, \alpha_2, .., \alpha_n$ are dependent.
In this case det $(\beta_1, \alpha_2, .., \alpha_n) = 0$, by theorem 13. So we have to show

$$\det(\beta_1 + \beta_2, \alpha_2, .., \alpha_n) = \det(\beta_2, \alpha_2, .., \alpha_n)$$

There are two possibilities : (*i*) β_1 is not dependent on $\alpha_2, .., \alpha_n$.

Then $\alpha_2, .., \alpha_n$ must be dependent. So, let $\alpha_2 = \sum_{k=3}^{n} d_k\alpha_k$. Therefore

$$\det(\beta_1 + \beta_2, \alpha_2, .., \alpha_n) = 0 = \det(\beta_2, \alpha_2, .., \alpha_n),$$

and the theorem is proved.

(*ii*) β_1 is dependent on $\alpha_2, .., \alpha_n$. So, let $\beta_1 = \sum_{k=2}^{n} d_k\alpha_k$. Therefore

$$\det(\beta_1 + \beta_2, \alpha_2, .., \alpha_n) = \det(\beta_1 + \beta_2 - \Sigma\, d_k\alpha_k, \alpha_2, .., \alpha_n)$$

$= \det(\beta_2, \alpha_2, .., \alpha_n)$, and the theorem is proved.

(2) $\beta_1, \alpha_2, .., \alpha_n$ are independent. Here β_2 must be dependent on $\beta_1, \alpha_2, .., \alpha_n$, because, by theorem 2, there cannot exist more than n

independent n-vectors. So let $\beta_2 = d_1\beta_1 + \sum_{k=2}^{n} d_k\alpha_k$. Therefore

$$\begin{aligned}
\det(\beta_1+\beta_2, \alpha_2, \ldots, \alpha_n) &= \det(\beta_1 + d_1\beta_1 + \Sigma d_k\alpha_k, \alpha_2, \ldots, \alpha_n)\\
&= \det(\beta_1 + d_1\beta_1, \alpha_2, \ldots, \alpha_n) = (1+d_1)\det(\beta_1, \alpha_2, \ldots, \alpha_n)\\
&= \det(\beta_1, \alpha_2, \ldots, \alpha_n) + \det(d_1\beta_1, \alpha_2, \ldots, \alpha_n)\\
&= \det(\beta_1, \alpha_2, \ldots, \alpha_n) + \det(d_1\beta_1 + \Sigma d_k\alpha_k, \alpha_2, \ldots, \alpha_n)\\
&= \det(\beta_1, \alpha_2, \ldots, \alpha_n) + \det(\beta_2, \alpha_2, \ldots, \alpha_n),
\end{aligned}$$

and the theorem is proved.

We now come to the general case. For the sake of definiteness, let $i = 1$. Then, by repeated application of the particular case proved above,

$$\begin{aligned}
\det A &= \det(c_1\beta_1 + \ldots + c_t\beta_t, \alpha_2, \ldots, \alpha_n)\\
&= \det(c_1\beta_1, \alpha_2, \ldots, \alpha_n) + \ldots + \det(c_t\beta_t, \alpha_2, \ldots, \alpha_n)\\
&= c_1 \det B_1 + \ldots + c_t \det B_t
\end{aligned}$$

Theorem 15. *Let A_{ik} be the determinants obtained from $\det A$ by replacing α_i by the unit vector ϵ_k, where ϵ_k is defined by (2), $k = 1, \ldots, n$. Then*

$$\sum_k a_{ik}A_{ik} = \det A \quad \text{and} \quad \sum_k a_{ik}A_{jk} = 0, \quad \text{if } i \neq j \tag{11}$$

Proof : The first formula follows from the last theorem together with the fact that $\alpha_i = \sum_k a_{ik}\epsilon_k$. For the second formula, suppose $i < j$. Since

$$\Sigma a_{ik}A_{ik} = \det(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_n) = \det A,$$

so

$$\Sigma a_{ik}A_{jk} = \det(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_i, \ldots, \alpha_n) = 0$$

Def. The determinants A_{ik} are called the *cofactors* of a_{ik} in $\det A$.

Note : From the first formula of this theorem it follows that a determinant can be expressed as a sum of products of the constituents of any row and their cofactors. As the cofactors A_{ik} do not contain any constituent of the ith row, it follows that a determinant can be expressed as a linear function of the constituents of any one of its rows, the coefficients being the cofactors of these constituents.

Theorem 16. $$\det A = \Sigma \pm a_{1i_1} a_{2i_2} \ldots a_{ni_n}, \tag{12}$$
where the sum has to be taken for the indices $i_1, \ldots, i_n$ such that $i_1 \ldots i_n$ shall denote a permutation of $12 \ldots n$. The + sign has to be taken before a term if $i_1 \ldots i_n$ is an even permutation of $1 \ldots n$ and − sign if an odd permutation of $1 \ldots n$.

(For properties of permutations, see later § 7)

Proof : From the first formula of the last theorem we get successively

$$\begin{aligned}\det A &= \sum_{i_1} a_{1i_1} \det(\epsilon_{i_1}, \alpha_2, \ldots, \alpha_n)\\ &= \sum_{i_1, i_2} a_{1i_1} a_{2i_2} \det(\epsilon_{i_1}, \epsilon_{i_2}, \alpha_3, \ldots, \alpha_n) = \ldots.\\ &= \sum_{i_1, \ldots, i_n} a_{1i_1} a_{2i_2} \ldots a_{ni_n} \det(\epsilon_{i_1}, \epsilon_{i_2}, \ldots, \epsilon_{i_n})\end{aligned}$$

Here in the summation each of the indices $i_1, \ldots, i_n$ takes values from 1 to n. But from property (c), theorem 12 and its corollary it follows that $\det(\epsilon_{i_1} \ldots \epsilon_{i_n}) = +1, -1$ or 0 according as $i_1 \ldots i_n$ is an even permutation, an odd permutation of $1 \ldots n$ or not all distinct. Hence the theorem.

Note : It would follow from this theorem that the function $\det A$ satisfying the properties (a), (b), (c) exists.

THEOREM 17. *If in* $\det A$, *the successive row-vectors are replaced by the successive column-vectors,* $\det A$ *remains unaltered.*

Proof : Let the matrix A be changed into the matrix $\bar{A}$ when the successive column-vectors of A are written as the successive row-vectors. Then, by the last theorem,

$$\det \bar{A} = \Sigma \pm a_{i_1 1} a_{i_2 2} \ldots a_{i_n n},$$

where + or − sign is taken according as $i_1 \ldots i_n$ is an even or an odd permutation of $1 \ldots n$. Now, since $i_1 \ldots i_n$ is a permutation of $1 \ldots n$, the factors in every term $a_{i_1 1} \ldots a_{i_n n}$ can be so arranged that their first indices $i_1, \ldots, i_n$ take the natural order $1 \ldots n$, the order of the second indices being then changed into $k_1 \ldots k_n$, say, which is the permutation inverse to $i_1 \ldots i_n$. So

$$\det \bar{A} = \Sigma \pm a_{1k_1} a_{2k_2} \ldots a_{nk_n}$$

But as two inverse permutations are both even or both odd, so

$$\det \bar{A} = \det A.$$

Note : It follows from this theorem that every theorem which holds for determinants with respect to its row-vectors also holds with respect to its column-vectors. Thus, in the properties (a), (b), (c) and in all subsequent theorems 12 – 16, we may replace the row-vectors $\alpha_1, \ldots, \alpha_n$ by the column-vectors $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ respectively.

THEOREM 18. *If in* $\det A$, *both the row and the column in which* a_{ik} *occurs are omitted, we obtain a determinant of* $n-1$ *rows and columns which is equal to* $(-1)^{i+k} A_{ik}$.

Proof : The equality of the two determinants may be seen to hold if both of them are developed by means of the formula (12).

Note : Since a cofactor A_{ik} in det A of n rows and columns is a determinant (with proper sign) of $n-1$ rows and columns, a cofactor in A_{ik} again is a determinant (with proper sign) of $n-2$ rows and columns. If every cofactor is thus reduced gradually, theorem 16 would follow from the first of the formulae (11).

Def. Let B be a $m \times n$ matrix and r be a positive integer which is less than or equal to the lesser of the two numbers m and n. If in B we omit (or strike out) $m-r$ rows and $n-r$ columns, we obtain an $r \times r$ matrix. The determinant of this matrix (with a + or − sign) is called a *minor of B of order r*.

From the above definition the following properties are easily seen to follow :

(1) If $m = n$, then (*i*) the minor of B of order n is $\pm$ det B, (*ii*) every cofactor in det B is a minor of order $n-1$ and (*iii*) if every minor of order $\leq n-1$ is zero, every minor of higher order is also zero.

(2) If $m \leq n$ and if there exists a minor of B of order m which is not zero, then the rank of B is m.

THEOREM 19. *If a matrix B has at least one minor of order r which is not equal to zero but every minor of higher order (if any) is equal to zero, then the rank of B is r.*

(This theorem is often taken as the definition of the rank of a matrix.)

Proof : Let

$$B = \begin{pmatrix} a_{11} \dots a_{1r} \dots a_{1s} \dots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{r1} \dots a_{rr} \dots a_{rs} \dots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{m1} \dots a_{mr} \dots a_{ms} \dots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{pmatrix}$$

and let the row-vectors of B be denoted by $\alpha_1, \dots, \alpha_r, \dots \alpha_m, \dots$ As the rank of a matrix is not altered by any permutation of rows or of columns, we suppose, without loss of generality, that the determinant formed out of the first r rows and columns is the minor M of order r which is different from zero. Therefore the rank of the matrix formed by the first r rows of B is r and so the row-vectors $\alpha_1, \dots, \alpha_r$ are independent. Let D be the determinant of $r+1$ rows formed by the first r rows and the mth row and the first r columns and the sth column of B. If $A_1, \dots, A_r, A_m$ are respectively the cofactors of $a_{1s}, \dots, a_{rs}, a_{ms}$ in D, then

$$D = a_{1s} A_1 + \dots + a_{rs} A_r + a_{ms} A_m$$

Now D is a minor of B of order $r+1$ and so, by hypothesis, $D = 0$, Further, $A_m = \pm M \neq 0$, by hypothesis. Therefore

$$a_{1s} A_1 + \dots + a_{rs} A_r + a_{ms} A_m = 0, \quad A_m \neq 0$$

This result holds if for s we write $1, \dots, r$, because the left-hand side then gives determinants with indentical columns which necessarily vanish. This result also holds if for s we write $r+1, r+2, \dots$, because this means that the sth column of B has been replaced by the $(r+1)$th, $(r+2)$th, ... columns for the formation of D and so the left-hand side gives minors of B of order $r+1$ which vanish by hypothesis. Therefore the above equations can be written as one equation in vectors as

$$\alpha_1 A_1 + \dots + \alpha_r A_r + \alpha_m A_m = 0, \qquad \text{where} \quad A_m \neq 0$$

As $A_1, \dots, A_r$, A_m are numbers, this equation shows that α_m is dependent on $\alpha_1, \dots, \alpha_r$. This remains true for $m = r+1, r+2, \dots$ Therefore $\alpha_1, \dots, \alpha_r$ form a basis of $R(B)$. Hence the rank of B is r.

COROLLARY. *If A is a square matrix and $\det A \neq 0$, then the rank of $A = n$. If $\det A = 0$, rank of $A < n$.*

As an application of a non-vanishing minor of the highest order, we give below the solution of a system of n independent linear equations in n unknowns in a form which is very often used. We take the case of $n = 3$, the method being perfectly general. Let the system be

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= a_0 \\ b_1x_1 + b_2x_2 + b_3x_3 &= b_0 \\ c_1x_1 + c_2x_2 + c_3x_3 &= c_0 \end{aligned} \qquad D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0$$

Multiply the equations respectively by the cofactors of a_1, b_1, c_1 in D and add. Similarly multiply the equations by the cofactors of a_2, b_2, c_2 in D and add and by the cofactors of a_3, b_3, c_3 in D and add. Then, by theorem 15 and note to theorem 17, we get the following solution :

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}, \tag{13}$$

$$\text{where } D_1 = \begin{vmatrix} a_0 & a_2 & a_3 \\ b_0 & b_2 & b_3 \\ c_0 & c_2 & c_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_1 & a_0 & a_3 \\ b_1 & b_0 & b_3 \\ c_1 & c_0 & c_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_2 & a_0 \\ b_1 & b_2 & b_0 \\ c_1 & c_2 & c_0 \end{vmatrix}$$

If $a_0 = b_0 = c_0 = 0$, the solution is $x_1 = x_2 = x_3 = 0$. These results agree with what is stated in the corollary of theorem 11. The rule by which the solution (13) is obtained is known as *Cramer's rule.*

5. Products of matrices and of determinants.

In order to define the product of two matrices, it will be advantageous to define initially what is known as the scalar product of two n-vectors

$$\alpha = (a_1, \ldots, a_n), \qquad \beta = (b_1, \ldots, b_n)$$

The *scalar product of* α *and* β is a number defined as

$$\alpha . \beta = a_1 b_1 + \ \ldots \ + a_n b_n \tag{14}$$

Now let A be an $m \times n$ matrix and B an $n \times p$ matrix. Let the row-vectors of A be denoted by $\alpha_1, \ldots, \alpha_m$ and the column-vectors of B by $\bar{\beta}_1, \ldots, \bar{\beta}_p$. Evidently the α's and the $\bar{\beta}$'s are all n-vectors. Form the mp scalar products

$$\alpha_i . \bar{\beta}_j, \qquad i = 1, \ldots, m \,;\, j = 1, \ldots, p$$

Then the product AB of the matrices A and B is *defined* as the $m \times p$ matrix

$$AB = \begin{pmatrix} \alpha_1 . \bar{\beta}_1 \ldots\ldots\ldots \alpha_1 . \bar{\beta}_p \\ \alpha_2 . \bar{\beta}_1 \ldots\ldots\ldots \alpha_2 . \bar{\beta}_p \\ . \quad . \quad . \quad . \quad . \quad . \\ \alpha_m . \bar{\beta}_1 \ldots\ldots\ldots \alpha_m . \bar{\beta}_p \end{pmatrix} \tag{15}$$

It is at once seen that for this definition, the product AB is not generally equal to the product BA. This is expressed by saying that the multiplication of matrices does not obey the *commutative law.*

In particular, let $A = (a_{ij})$ and $B = (b_{jk})$. If $AB = (f_{ik})$, it follows from (3) and (15) that

$$f_{ik} = \sum_{s=1}^{n} a_{is} \, b_{sk}$$

Let now $C = (c_{kl})$ be a $p \times q$ matrix. If the $m \times q$ matrix $(AB)C = (g_{il})$, then we get

$$g_{il} = \sum_{t=1}^{p} f_{it} \, c_{tl} = \sum_{s=1}^{n} \ \sum_{t=1}^{p} a_{is} \, b_{st} \, c_{tl}$$

It can now be directly verified that if $A(BC) = (h_{il})$, then $g_{il} = h_{il}$ and therefore

$$(AB)C = A(BC)$$

This is expressed by saying that the matrix multiplication obeys the *associative law*.

Def. A *diagonal matrix* is a square matrix whose constituents not situated on the main diagonal are all zero. Thus, if (d_{ij}) is a diagonal matrix, then $d_{ij} = 0$ for $i \neq j$.

An *elementary matrix*, denoted by (say) $E_{st}(a)$, $s \neq t$, is a square matrix, which is defined thus : Let $E_{st}(a) = (e_{ij})$. Then

$$e_{ii} = 1 \ (i = 1, 2, \ldots), \quad e_{st} = a, \quad e_{ij} = 0 \text{ for } i \neq j \text{ and } (i, j) \neq (s, t).$$

Thus $$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \quad \text{and} \quad E_{23}(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

are diagonal and elementary matrices respectively.

Let A be an $n \times n$ matrix whose row- and column-vectors are denoted, as before, by α_i and $\bar{\alpha}_i$ respectively, $i = 1, \ldots, n$. Also, let D and $E_{st}(a)$ be diagonal and elementary matrices which have the same number of rows as A, where the successive diagonal constituents of D are $d_1, \ldots, d_n$. Then the following results can be easily seen :

(*i*) DA (or AD) is the matrix obtained by multiplication of the rows (or columns) of A by $d_1, \ldots, d_n$ respectively.

(*ii*) $E_{st}(a)A$ (or $AE_{st}(a)$) gives a row-addition (or column-addition) in A by which α_s (or $\bar{\alpha}_t$) of A is replaced by $\alpha_s + a\alpha_t$ (or $\bar{\alpha}_t + a\bar{\alpha}_s$).

(*iii*) $\det DA = \det AD = \det D \det A = d_1 \ldots d_n \det A,$
$\det E_{st}(a)A = \det AE_{st}(a) = \det A.$

THEOREM 20. *A square matrix A can be transformed into a diagonal matrix by row-additions and column-additions.*

Proof : If the matrix $A = (a_{ij})$ is a zero-matrix, it is already a diagonal matrix. If not, we can suppose $a_{11} \neq 0$. (For, if $a_{11} = 0$, we can, by row- and column-additions, so arrange that the constituent in the first row and column becomes different from zero.) By suitable row-additions the constituents in the first column, other than a_{11}, can be made zero and then by suitable column-additions, the non-zero constituents of the first row can be made zero without altering the first column. If the resulting matrix is not a diagonal matrix by now, we can treat the second row and column in the same way as before. This procedure can be repeated until A becomes a diagonal matrix.

As every row-addition (or column-addition) means a multiplication by an elementary matrix from the left (or right), it follows from the above theorem that

(*iv*) $$A = P_1 D P_2,$$

where P_1 and P_2 are products of elementary matrices and D is a diagonal matrix.

THEOREM 21. *If A and B are square matrices with the same number of rows, then*

$$\det AB = \det A \det B. \tag{16}$$

Proof : By (*iv*) and (*iii*),

$\det A = \det P_1 D P_2 = \det P_1 D = \det D P_2 = \det D = d_1 \ldots d_n.$
Therefore applying the results (*i*) to (*iv*) given above, we get

$$\det AB = \det P_1 D P_2 B = \det D P_2 B$$
$$= d_1 \ldots d_n \det P_2 B = d_1 \ldots d_n \det B = \det A \det B.$$

Note : It is to be noticed that since a determinant is a number, the order of factors is immaterial in a determinant product. As a matter of fact, the distinction between row-vectors and column-vectors of a determinant is unnecessary in view of theorem 17.

Def. If the successive rows and columns of an $m \times n$ matrix A are interchanged, an $n \times m$ matrix is obtained which is called the *transposed* of A and is denoted by A^{T}. Thus the ith row-vector of A is the ith column-vector of A^{T}, and vice versa. It can be easily verified from (15) that $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$. If A is a square matrix and $A = A^{\mathrm{T}}$, then A is called a *symmetric matrix*. The special diagonal matrix

$$I = \begin{pmatrix} 1 & 0 \ldots\ldots\ldots 0 \\ 0 & 1 \ldots\ldots\ldots 0 \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 & 0 \ldots\ldots\ldots 1 \end{pmatrix}$$

is called a *unit matrix*. A square matrix A is called an *orthogonal matrix* if $AA^{\mathrm{T}} = I$. Therefore if $A = (a_{ij})$ is an orthogonal matrix, we get from (15) and (16) that

$$\sum_k a_{ki} a_{kj} = \sum_k a_{ik} a_{jk} = \left[\begin{array}{ll} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{array} \right. \quad \text{and } (\det A)^2 = 1$$

Given a square matrix A, if there exists a matrix B such that $AB = I$, and therefore also $BA = I$, then B is called the *inverse* of A. It therefore follows that the transposed of an orthogonal matrix is its inverse. The inverse of a matrix when it exists is obtained in the next article.

6. Linear transformations. The equations

$$\begin{matrix} y_1 = a_{11}x_1 + \dots + a_{1n}x_n \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ y_n = a_{n1}x_1 + \dots + a_{nn}x_n \end{matrix} \qquad A = \begin{pmatrix} a_{11}\dots a_{1n} \\ \dots \\ a_{n1}\dots a_{nn} \end{pmatrix} \qquad (17)$$

define a *linear transformation* transforming the n-vectors $\xi = (x_1, \dots, x_n)$ into the n-vectors $\eta = (y_1, \dots, y_n)$. To every vector ξ there corresponds uniquely a (transformed) vector η. This correspondence may be denoted by $\xi \longrightarrow \eta$. In particular, referring to (2).

$$0 \longrightarrow 0,\ e_1 \longrightarrow (a_{11}, \dots, a_{n1}), \dots\dots,\ e_n \longrightarrow (a_{1n}, \dots, a_{nn})$$

Now let $\xi_1 \longrightarrow \eta_1$ and $\xi_2 \longrightarrow \eta_2$ by the linear transformation (17). Then

$$\xi_1 + \xi_2 \longrightarrow \eta_1 + \eta_2 \qquad \text{and} \qquad c\xi_1 \longrightarrow c\eta_1, \text{ where } c \text{ is a number.}$$

It therefore follows from theorem 3 that if the ξ-vectors form a vector space, then the η-vectors also form a vector space. That is, a linear transformation transforms a vector space into a vector space.

Suppose we have a second linear transformation transforming the vectors η into the vectors $\zeta = (z_1, \dots, z_n)$ defined by

$$\begin{matrix} z_1 = b_{11}y_1 + \dots + b_{1n}y_n \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ z_n = b_{n1}y_1 + \dots + b_{nn}y_n \end{matrix} \qquad B = \begin{pmatrix} b_{11}\dots b_{1n} \\ \dots \\ b_{n1}\dots b_{nn} \end{pmatrix}$$

Applying the second transformation after the first we get a linear transformation transforming the vectors ξ into the vectors ζ. The equations of the resultant transformation is obtained as follows:

We have $\qquad y_i = \sum_j a_{ij}x_j, \qquad z_k = \sum_i b_{ki}y_i$

Therefore $\qquad z_k = \sum_{i,j} b_{ki}a_{ij}x_j,$

or $\qquad z_k = \sum_j p_{kj}x_j, \qquad \text{where} \qquad p_{kj} = \sum b_{ki}a_{ij}$

These are the required equations of the resultant linear transformation.

The matrix of the resultant transformation is therefore given, from (15), by

$$P = \begin{pmatrix} p_{11} \cdots p_{1n} \\ \cdot \quad \cdot \quad \cdot \\ p_{n1} \cdots p_{nn} \end{pmatrix} = \begin{pmatrix} \Sigma\, b_{1i}\, a_{i1} \ldots \Sigma\, b_{1i}\, a_{in} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \Sigma\, b_{ni}\, a_{i1} \ldots \Sigma\, b_{ni}\, a_{in} \end{pmatrix} = BA$$

Accordingly, if a linear transformation T_1 is followed by a linear transformation T_2 and the resultant is a linear transformation T_3, it is usual to write $T_2T_1 = T_3$. The reason behind the rule of matrix multiplication as given in (15), which might have appeared arbitrary, is now apparent.

It is often convenient to express an n-vector $\xi = (x_1, \ldots, x_n)$ in the form of a matrix as

$$(x) = \begin{pmatrix} x_1\; 0 \ldots 0 \\ \cdot \quad \cdot \quad \cdot \\ x_n\; 0 \ldots 0 \end{pmatrix}$$

In view of rule (15), the above two component linear transformations can then be written as

$$(y) = A(x) \quad \text{and} \quad (z) = B(y)$$

Combining the two we get

$$(z) = BA(x)$$

This formula agrees with the equations of the resultant transformation as given above.

We have seen that the linear transformation (17) establishes a correspondence which may be denoted by $\xi \longrightarrow \eta$. Under what condition does the inverse correspondence $\eta \longrightarrow \xi$ exist, so that there may be a one-to-one correspondence, denoted by $\xi \longleftrightarrow \eta$, between the vectors ξ and the vectors η ?

Multiply the successive equations (17) respectively by the cofactors $A_{1k}, \ldots, A_{nk}$ of the constituents $a_{1k}, \ldots, a_{nk}$ of the kth column in det A and add. Then, by (11) and note to theorem 17, we get

$$x_k \det A = A_{1k}y_1 + \ldots + A_{nk}y_n, \;\; k = 1, \ldots, n \tag{18}$$

It therefore follows that the *inverse* of the linear transformation (17) exists and is given by (18) if and only if det $A \neq 0$, *i.e.*, the rank of A is n.

That is to say, the one-to-one correspondence as proposed exists if and only if $\det A \neq 0$. When it exists, the matrix of the linear transformation (18), denoted by A^{-1}, is given by

$$A^{-1} = \begin{pmatrix} \dfrac{A_{11}}{\det A} \cdots \dfrac{A_{n1}}{\det A} \\ \cdots\cdots\cdots\cdots\cdots \\ \dfrac{A_{1n}}{\det A} \cdots \dfrac{A_{nn}}{\det A} \end{pmatrix} \tag{19}$$

It may be verified by rule (15) that

$$AA^{-1} = A^{-1}A = I \tag{20}$$

It therefore follows that the inverse of A is A^{-1} and this relation is a reciprocal one. As the matrix of the resultant of the transformations (17) followed by its inverse (18) is a unit matrix, the resultant leaves every vector unaltered.

When $\det A \neq 0$, the transformation (17) and its inverse (18) can be written in the matrix form as

$$(y) = A(x) \quad \text{and} \quad (x) = A^{-1}(y) \tag{21}$$

The significance of the rank of A is given by the following theorem :

THEOREM 22. *By the linear transformation (17), a vector space of rank n is transformed into a vector space of rank equal to the rank of A.*

Proof : Let the vectors $\xi = (x_1, \ldots, x_n)$ form the vector space W and the vectors $\eta = (y_1, \ldots, y_n)$ form the vector space V. As W is supposed to be of rank n, so $x_1, \ldots, x_n$ take independently all values. A basis of W is therefore given by the vectors $e_1, \ldots, e_n$ defined by (2). Accordingly $\xi = \Sigma x_k e_k$.

Now if the column-vectors of A are denoted by $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$, then $e_k \to \bar{\alpha}_k$ by (17). So $\xi \to \Sigma x_k \bar{\alpha}_k$. Therefore the η-vectors are given by $\eta = \Sigma x_k \bar{\alpha}_k$. As $x_1, \ldots, x_n$ take all values independently, the vector space V is generated by $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$. But the number of independent column-vectors is the rank of A. Hence the rank of V is equal to the rank of A.

Regarding the rank of the product of matrices arising out of the resultant transformation we have the following theorem :

THEOREM 23. *Let A and B be two $n \times n$ matrices of rank r and s respectively and let the rank of AB be p. Then p can exceed neither r nor s ; and if r (or s) is equal to n, then p is equal to s (or r).*

Proof : Let the row-vectors of B be denoted by $\beta_1, \ldots, \beta_n$ and the row-vectors of AB be denoted by $\gamma_1, \ldots, \gamma_n$. Also, let $\phi_1, \ldots, \phi_s$ be a basis of $R(B)$ and $\psi_1, \ldots, \psi_p$ be a basis of $R(AB)$. It then follows from the theory of vector spaces and the definition of matrix multiplication that every ψ_i is a linear combination of the γ's and every γ_i is a linear combination of the β's and every β_i is a linear combination of the ϕ's. Therefore every γ_i is linearly dependent on the ϕ's. That is, the rank of $R(AB)$ cannot exceed the rank of $R(B)$. In a similar manner it can be seen that the rank of $C(AB)$ cannot exceed the rank of $C(A)$. But as the rank of $R(M)$ = the rank of $C(M)$ = the rank of M for every matrix M, so p is less than or equal to the minimum of r and s.

Again, if the rank of A is n, A^{-1} exists. Therefore, as $B = A^{-1}AB$, so $s \leq p$. But $p \leq s$. Hence $p = s$. Similarly if $s = n$, then $p = r$. Hence the theorem.

7. Groups. The word 'group' is used in a specialised abstract sense. We shall do no more here than give a formal definition of this concept and illustrate it by a few examples.

Let G be set of *elements* of any kind. The word 'element' is used in the broad sense to denote any kind of entity. The elements belonging to G may be finite or infinite in number. Let us assume that there exists an *operation* by means of which every pair of elements of G can be composed to form an element of G. That is to say, if the elements of G are denoted by $a, b, c, \ldots$ and the operation by '$\circ$', then $a \circ b$ is an element of G. In this composition, the elements a and b are not necessarily distinct, nor are the elements $a \circ b$ and $b \circ a$ of G necessarily the same. Then given the operation, the set G is said to form a *group* if it satisfies the following four conditions :

(1) Every ordered pair of elements of G can be composed to form a unique element of G. That is, $a \circ b$ is unique for every ordered pair a, b. This is expressed by saying that G is closed for the operation.

(2) The operation is associative. That is, for every triad a, b, c,

$$(a \circ b) \circ c = a \circ (b \circ c)$$

(3) There exists in G a unique element, say e, called the *unit* element having the property

$$a \circ e = e \circ a = a,$$

for every element a of G.

(4) Corresponding to every element a of G, there exists in G a unique element, say a', called the *inverse* of a, having the property

$$a \circ a' = a' \circ a = e.$$

When the group G has the property that $a \circ b = b \circ a$ for every pair of elements a, b, then the group is called *commutative* or *Abelian*. If S is a subset of a group G and if the above four conditions are satisfied for S with respect to the same operation as defined in G, then S is called a *subgroup* of G. Obviously, the unit element of a group forms by itself a subgroup; this subgroup is rather trivial. As a simple illustration, consider the set of all integers:

$$0, \pm 1, \pm 2, \pm 3, \ldots .$$

When the operation is taken as addition, this set form a commutative group and the set of even integers is a subgroup of this group; the unit element is the number 0 and the inverse of an integer a is $-a$. But if the operation is chosen as multiplication, the set does not form a group. On the other hand, the set of all non-zero rational numbers form a multiplicative, but not an additive, group; the unit element is 1 and the inverse of a rational number a of the set is a^{-1}.

Next take either the set of all rational numbers or the set of all real numbers or the set of all complex numbers, and consider all $n \times n$ matrices formed with the numbers of this set. We then have the following theorem:

THEOREM 24. *The set of all $n \times n$ matrices whose determinants are not zero form a group, the operation being the matrix multiplication.*

Proof: That the conditions (1) and (2) of a group are satisfied by these matrices can be seen from (15) and its sequel. And conditions (3) and (4) are satisfied by virtue of (20).

As another illustration of a group, consider the set of all permutations of n objects. A permutation can be regarded as a transformation by which the objects are interchanged among themselves. That is to say, if the objects are denoted by the digits $1, \ldots, n$, then a permutation $a_1 \ldots a_n$ of $1 \ldots n$ can be expressed by the notation

$$P = \begin{pmatrix} 1 \ldots k \ldots n \\ a_1 \ldots a_k \ldots a_n \end{pmatrix} = \begin{pmatrix} k \\ a_k \end{pmatrix}, \tag{22}$$

where the n objects are ordered in the second row in such a manner that below every k in the first row is written a_k in the second row into which k is transformed. It therefore follows that the permutation P remains unaltered by interchange of columns. It is obvious that the notation (22) does not denote any matrix.

Let now $b_1 \ldots . b_n$ be another permutation Q of $1 \ldots . n$. As k, a_k, b_k can take all objects $1, \ldots, n$, Q can be denoted by

$$Q = \begin{pmatrix} k \\ b_k \end{pmatrix} = \begin{pmatrix} a_k \\ b_{a_k} \end{pmatrix}$$

If P is applied first and then Q, k is transformed into b_{a_k}. The resultant permutation thus obtained is called the *product* of P and Q and is denoted by

$$QP = \begin{pmatrix} k \\ b_{a_k} \end{pmatrix} \tag{23}$$

Evidently the product is associative but not commutative. Given P, if Q is such that the product is the identical permutation or the *identity*

$$J = \begin{pmatrix} 1 \ldots . n \\ 1 \ldots . n \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix},$$

then Q is called the *inverse* of P and is denoted by P^{-1}, and, with reference to (22), it can be written as

$$P^{-1} = \begin{pmatrix} a_1 \ldots . a_k \ldots . a_n \\ 1 \ldots . k \ldots . n \end{pmatrix} = \begin{pmatrix} a_k \\ k \end{pmatrix}$$

Thus, for every permutation P, we get

$$PJ = JP = P, \quad PP^{-1} = P^{-1}P = J \tag{24}$$

Suppose that in a permutation we find that an object a_1, chosen arbitrarily, is transformed into a_2, a_2 into $a_3, \ldots, a_{m-1}$ into a_m and a_m into a_1, $m \leq n$. The the sequence $a_1, \ldots, a_m$ is a cyclic sequence or a cycle consisting of m objects. It is evident that every permutation has at least one cycle. A permutation which has just one cycle of $m > 1$ objects is called a *cyclic permutation* and is denoted by

$$\begin{pmatrix} a_1 \ldots . a_{m-1} & a_m & a_{m+1} \ldots . a_n \\ a_2 \ldots . a_m & a_1 & a_{m+1} \ldots . a_n \end{pmatrix} = (a_1, \ldots, a_m)$$

In particular, if $m = 2$, a cyclic permutation is called a *transposition*.

LEMMA 1. *Every permutation P can be uniquely expressed as a product of a finite number of cyclic permutations and (or) cycles of one object such that no two cycles have a common object.*

Proof : Let an object a_1 determine the cycle $a_1, \ldots, a_{m_1}$. Then, if possible, let an object b_1, not contained in this cycle, determine the cycle $b_1, \ldots, b_{m_2}$. Proceeding in this manner P can, after a finite number of steps, be expressed as

$$P = (a_1, \ldots, a_{m_1})(b_1, \ldots, b_{m_2}) \ldots (d_1, \ldots d_{m_r})$$

A cycle is said to be even or odd according as the number of objects belonging to it is even or odd. Let a permutation P be expressed as a product of cyclic permutations as in lemma 1. Then P is said to be *even* or *odd* according as the number of even cycles composing it is even or odd. Every permutation is therefore either even or odd. Thus J is an even permutation and a transposition is an odd permutation. P and P^{-1} are both even or both odd. We state, without proof, the following lemma :

LEMMA 2. *The product of two even permutations and of two odd permutations are even permutations. The product of an even and an odd permutation is an odd permutation.*

In view of formulae (23), (24) and lemma 2, we arrive at the following theorem :

THEOREM 25. *The set of all permutations of n objects form a group of which the subset of even permutations form a subgroup.*

These two groups are called the *symmetric group* and the *alternating group* respectively.

A group consisting of a finite number of elements is called a *finite group*, and the number of elements of a finite group is called its *order*. Thus the symmetric and alternating groups are finite and their orders are respectively $n!$ and $n!/2$. The group of matrices of theorem 24 whose constituents are rational, real or complex numbers is not a finite group

Let the operation in a group G be called multiplication. If every element of a subgroup S of G is multiplied from the left by an element a of G we get what is called a *left coset* aS which is not necessarily a subgroup. Similarly for a *right coset* Sb. It can however be proved that aSa^{-1} is a subgroup called a *conjugate* of S. If the subgroup S is such that $aSa^{-1} = S$ for every element a of G, then S is called a *normal subgroup* of G. As an example we may state, without proof, that the alternating group is a normal subgroup of the symmetric group.

PLANE GEOMETRY

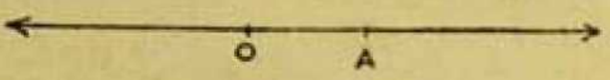
CHAPTER I

VECTORS AND ANGLES

1. Points and vectors on a straight line. Let a straight line g be taken. If on g we take an arbitrary but fixed point O, all points of g situated on the same side of O constitute a *half-ray* emanating from O. The point O therefore divides g into two half-rays, g', g'', say. We measure distances along the two half-rays with the help of an arbitrary but fixed unit segment u. If A is a point of g, we measure the distance $|OA| = |x|$ by the ratio of the segments OA and u, where $|x| = 0$ if A coincides with O and is a positive number in every other case. To every real positive number there corresponds, on the other hand, one point of g' and one point of g''.

In order to establish a one-to-one correspondence between the points of g and real numbers, we further introduce positive and negative *directions* along g and choose one of the half-rays, say g', as giving the positive direction while the other the negative direction. We assign

$x = \quad |x| = 0$, if A coincides with O
$x = \quad |x| > 0$, if A is a point of g'
$x = -|x| < 0$, if A is a point of g''.



The real number x is then said to be the *coordinate* of the point A.

The one-to-one correspondence thus established depends on three arbitrarily chosen items : the point O (that is, the point whose coordinate is zero), the unit segment u and one of the half-rays giving either the positive or the negative direction. And when these items are fixed, the one-to-one correspondence is uniquely determined. The distance between two points A_1, A_2 of g, whose coordinates are x_1, x_2, is then given by

$$|A_1 A_2| = |x_1 - x_2|$$

Let A_1 be an arbitrary point of g distinct from O and x_1 its coordinate. Then A_1 defines two half-rays g_1', g_1'', where g_1' is composed of points with coordinates $x > x_1$ and g_1'' of points $x < x_1$. Whatever be the position of A_1, the half-rays g' and g_1' as also g'' and g_1'' have common points ; but either g'' has no point in common with g_1' or g' has no point in common with g_1''. In either case, the half-rays g' and g_1' as also g'' and g_1'' are said to have the same directions.

Any two points A, B of g determine a segment AB or BA. The points A, B are the extremities of the segment, the points lying between A and B are points within the segment and the *length* of the segment is the distance $| AB |$. A segment becomes a *directed segment* when we consider one of the extremities as the starting point and the other as the end point, and consider a direction from the starting point to the end point. The directed segment whose extremities are A, B and whose direction is from A to B shall be denoted by the notation $\overline{AB}$. The directed segments $\overline{AB}$ and $\overline{BA}$ have the same two extremities, the same length but opposite directions.

Let A_1, A_2 be two points of g whose coordinates are x_1, x_2 respectively. Then the number $x_2 - x_1$ is said to be the *coordinate* of $\overline{A_1A_2}$ and therefore $x_1 - x_2$ the coordinate of $\overline{A_2A_1}$. Accordingly, if the coordinate of a directed segment and that of one extremity are given, the coordinate of the other extremity becomes known. Directed segments of g having the same coordinate represent the same *vector*. A vector is therefore the class of all directed segments of g having the same coordinate and this coordinate is also used as the coordinate of the vector. There is no harm to denote a vector by any one of the notations that are used to denote the directed segments representing it. Thus, if A_3, A_4 are two other points of g with coordinates x_3, x_4 respectively, we shall write, as vectors,

$$\overline{A_1A_2} = \overline{A_3A_4}, \text{ if } x_2 - x_1 = x_4 - x_3$$

A vector therefore possesses a length and a direction, the length being that of any one of the directed segments representing it. A vector is accordingly uniquely determined by its coordinate. We may speak of *fixing* a vector $\overline{A_1A_2}$ at a point B_1 of g; this means, constructing the directed segment $\overline{B_1B_2}$ such that $\overline{B_1B_2} = \overline{A_1A_2}$.

Let A, B, C be three points of g. Then, as distances, either

$$| AC | = | AB | + | BC | \text{ or } | AC | = | AB | - | BC |$$

But, as vectors, we shall always have

$$\overline{AC} = \overline{AB} + \overline{BC}$$

It follows that the coordinate of the sum of two vectors is the sum of the coordinates of the summands. From this property again follows the commutative law of addition of vectors :

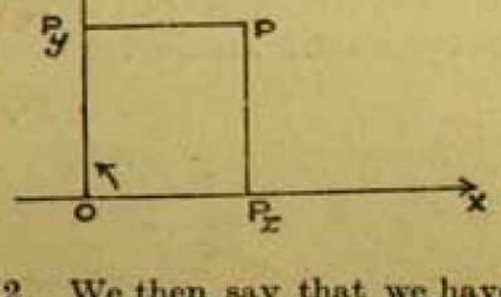
$$\overline{AB} + \overline{BC} = \overline{BC} + \overline{AB}$$

If v_i are the coordinates of $\overline{A_{i-1}A_i}$, for $i=1, 2, \ldots\ldots, n$, the coordinate of $\overline{A_0A_n}$ is $v_1+v_2+\cdots\cdots+v_n$ and $|A_0A_n| = |v_1+v_2+\cdots\cdots+v_n|$.

Let a be an arbitrary but fixed real number and g be transformed into itself in such a manner that every point P of g with coordinate x is transformed into a point P' of g with coordinate $x' = x+a$. This transformation is said to be a *displacement*. Any directed segment with coordinate x_2-x_1 will therefore be transformed into a directed segment with coordinate $x_2'-x_1' = (x_2+a)-(x_1+a) = x_2-x_1$. Hence, a displacement transforms a directed segment into a directed segment representing the same vector; that is to say, vectors will remain *unaltered* by the displacement. So, the lengths of segments will also remain unaltered. The two half-rays consisting of points $x > x_1$ and $x < x_1$ will be transformed into the half-rays of points $x > x_1+a$ and $x < x_1+a$. Hence the directions of half-rays remain unaltered. On the other hand, *every transformation of a straight line into itself which does not alter the lengths of segments and directions of half-rays is a displacement*. For, the point O is transformed into a certain point O' with coordinate a, say, an arbitrary point B with coordinate b is transformed into a point B' with the conditions that $|OB| = |O'B'|$ and the half-ray emanating from O on the same side as B has the same direction as the half-ray emanating from O' on the same side as B'; hence B' has the coordinate $a+b$. As this holds for every point B of the straight line, the transformation is a displacement. Thus, every point is displaced in a fixed direction through a fixed distance.

We have therefore two interpretations of the notion of a vector on a straight line. Firstly, a vector is given by a real number, its cordinate; and secondly, a vector can be interpreted as a displacement.

2. Points and vectors in the plane. In a given plane let any two arbitrary but fixed straight lines, called the x- and the *y-axes of coordinates*, be taken intersecting one another in a point O, called the *origin*. We shall, for the sake of simplicity, suppose that the axes are orthogonal. The origin divides each of the axes into two half-rays; the choice of the positive half-rays along the axes is, however, arbitrary. After the choice has been made, let us imagine the positive half-ray of x-axis to rotate about the origin in the positive sense, which, as usual, we shall suppose to be counter-clockwise, through an angle $\pi/2$. We then say that we have



a *right-handed* system of axes if, after rotation, the positive half-rays coincide ; otherwise, the system is *left-handed*. We shall always assume that our system is right-handed.

After having chosen, in an arbitrary manner, two congruent unit segments for the two axes, it is seen that to every point of an axis there corresponds a real number ; and conversely, to every real number there corresponds one point on each axis. Let P be any point of the plane. If through P parallels to the axes be drawn to meet the x- and the y- axes in the points P_x and P_y respectively, then the position of P is uniquely determined by the coordinate x of P_x and the coordinate y of P_y. So, to every point P of the plane there corresponds an *ordered* pair of numbers (x, y), called its coordinates, and conversely. If, in particular, the first (or the second) of these numbers is equal to zero, P is situated on the y- (or the x-) axis. We shall represent a point by its coordinates taken in brackets and write

$$P = (x, y)$$

The distance between the two points

$$P = (x_0, y_0), \quad \text{and} \quad O = (0, 0)$$

is then given by

$$| OP | = | \sqrt{(x^2+y^2)} | \tag{1.1}$$

As in the last article, we introduce directed segments in the plane by taking one of the extremities as the starting point and the other as the end point of every segment of every straight line in the plane. Let $P = (x_1, y_1)$ be the starting point and $Q = (x_2, y_2)$ the end point. By projecting $\overline{PQ}$ orthogonally on the x- and y- axes we obtain two directed segments $\overline{P_xQ_x}$ and $\overline{P_yQ_y}$ of the two axes respectively. Conversely, to every pair of directed segments of the two axes there corresponds one and only one directed segment in the plane. The coordinates of $\overline{P_xQ_x}$ and $\overline{P_yQ_y}$ form an ordered pair of numbers which are said to be the coordinates of $\overline{PQ}$. Directed segments of the plane having the same coordinates form a class which, as in the last article, is called a vector. A vector has the same coordinates as those of any one of the directed segments representing it, and there is no harm to denote the vector by any one of the notations that are used for these directed segments and to write, as vectors,

$$\overline{PQ} = (x_2 - x_1, y_2 - y_1) \tag{1.2}$$

and $\overline{P'Q'} = \overline{PQ}$ if $\overline{P'Q'}$ has also the coordinates (1.2). There is thus a one-to-one correspondence between the ordered pairs of real numbers and the vectors of the plane. As in the last article, we may speak of fixing a vector $\overline{PQ}$ at a given point P' ; this means, finding out the point Q'

such that $\overline{P'Q'} = \overline{PQ}$. So if $P'=(x', y')$, then $Q'=(x'+x_2-x_1, y'+y_2-y_1)$. This fixing can be made in the following two steps :

$$\overline{P''Q''} = \overline{PQ},$$

where $\qquad P'' = (x_1, y')$ and so $Q'' = (x_2, y'+y_2-y_1)$,

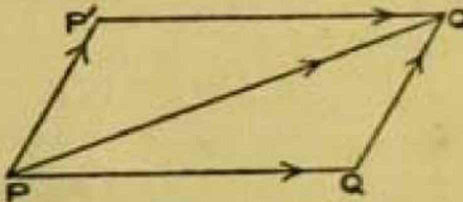
and $\qquad \overline{P'Q'} = \overline{P''Q''}$.

As $PQQ''P''$ and $P''Q''Q'P'$ are parallelograms, so $PQQ'P'$ is a parallelogram. The directed segments representing the same vector are therefore of equal length, parallel to one another and have the same direction. The geometrical meaning of fixing $\overline{PQ}$ at P' is therefore the finding out of the fourth vertex Q' of the parallelogram $PQQ'P'$.

Fixing $\overline{PQ}$ at the origin O, we obtain the fourth vertex Q^*, where Q^* has the coordinates (x_2-x_1, y_2-y_1). As the segments PQ and OQ^* have the same length, it follows from (1.1) that

$$|PQ| = |\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2\}}|$$

Let two arbitrary vectors be represented by the directed segments $\overline{PQ}$ and $\overline{QQ'}$, so that the end point of one is the starting point of the other. Then, the sum of these two vectors is *defined* as the vector represented by the directed segment $\overline{PQ'}$. Thus, the law of parallelogram holds for the addition of vectors; that is, if $PQQ'P'$ is a parallelogram of which PQ and PP' are adjacent sides, then, as vectors,



$$\overline{PQ} + \overline{PP'} = \overline{PQ'}$$

As in §1, the commutative law of addition holds. Also, it follows from (1.2) that the two coordinates of the sum of two vectors are respectively equal to the sum of the corresponding coordinates of the summands.

We shall often denote vectors by Greek letters. Let the vectors α_i have the coordinates (a_i, b_i), $i=1,2, \ldots .n$, and let them be represented by the directed segments $\overline{A_0A_1}, \overline{A_1A_2}, \ldots, \overline{A_{n-1}A_n}$, so that $A_0A_1 \ldots A_n$ is a broken line. Then, for the sum of the vectors,

$$\sum \alpha_i = \overline{A_0A_n} = (\sum a_i, \sum b_i).$$

The broken line is closed if and only if the vector $\overline{A_0A_n}$ is the *zero-vector* with coordinates (0, 0), *i.e.*, if and only if $\sum a_i = \sum b_i = 0$.

Let $\alpha = (a_1, a_2)$ be an arbitrary but fixed vector and let the plane be transformed into itself in such a manner that every point P is transformed into a point P', where $\overline{PP'} = \alpha$. This transformation is said to be a *parallel displacement*. Its analytic expression is

$$(x, y) \rightarrow (x + a_1, y + a_2).$$

If, by this transformation, a point Q is transformed into Q', then $PQQ'P'$ is a parallelogram. Hence, the arbitrary directed segment $\overline{PQ}$ is transformed into $\overline{P'Q'}$ representing the same vector. Accordingly, *vectors remain unaltered by parallel displacement*. On the other hand, if by an arbitrary transformation of the plane, vectors remain unaltered and P is transformed into P', then an arbitrary point Q has to be transformed into Q', where $PQQ'P'$ is a parallelogram ; hence the transformation is a parallel displacement. Thus, *if by a transformation of the plane no vector is altered, the transformation is a parallel displacement*. To every vector α there corresponds the parallel displacement *generated* by α. On the other hand, no vector is altered by any parallel displacement. This double connection between vector and parallel displacement should be noticed. By the parallel displacement generated by $\alpha = \overline{PQ}$, the straight line PQ is transformed into itself, and this transformation is a displacement considered in §1.

Given any real number λ, we *define* the product of λ and any vector $\alpha = (a_1, a_2)$ as the vector $\lambda\alpha$ having coordinates $(\lambda a_1, \lambda a_2)$. Its length is given by

$$|\lambda\alpha| = |\sqrt{\{(\lambda a_1)^2 + (\lambda a_2)^2\}}| = |\lambda| \, |\sqrt{(a_1^2 + a_2^2)}|$$

or,

$$|\lambda\alpha| = |\lambda| \, |\alpha|$$

If $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = \alpha$, then $\sum \alpha_i = n\alpha$.

For $\lambda = 0$, $O\overline{AB}$ becomes the zero-vector O ; for $\lambda = -1$, $(-1)\,\overline{AB} = \overline{BA}$. Therefore, as $\overline{AB} + \overline{BA} = O$, the addition of $(-1)\,\overline{AB}$ means the subtraction of $\overline{AB}$. We shall accordingly use the notation

$$\overline{AB} = -\overline{BA}$$

The vector $\lambda\alpha$ is therefore parallel to α ; its direction is the same or the opposite as that of α according as λ is positive or negative and its length is $|\lambda|\,|\alpha|$. The notion of product of a number and a vector is therefore independent of the choice of the coordinate system.

From the above consideration it follows that *for addition (subtraction) of vectors, we add (subtract) their coordinates, and for multiplication of a vector by a number we multiply its coordinates by that number.*

3. Angle. A given angle is bounded by two half-rays emanating from the same point. An angle is measured between two half-rays, from one to the other. The mersuring is considered positive in the counter-clockwise sense (of rotation) and negative in the opposite sense. Let $g_0, g_1, \cdots, g_n$ be a number of given half-rays emanating from the same point O and let a circle with centre O and radius unity be drawn ; the length of the circumference is then 2π. Also, let $\widehat{g_ig_k} = -\widehat{g_kg_i}$ be the length of the shortest arc measured from g_i to g_k, on the circumference of the circle, taken with the positive or the negative sign according as the sense is counter-clockwise or clockwise.

Then

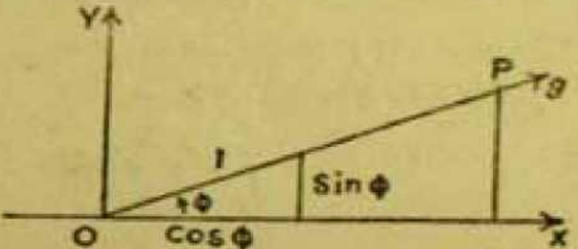
$$\widehat{g_0g_n} = \sum_{i=0}^{n-1} \widehat{g_ig_{i+1}} + 2k\pi,$$

where k is an integral number which, in some cases, is zero and, in other cases, positive or negative. The arc $\widehat{g_ig_k}$ is uniquely defined except in the case where g_i and g_k are different half-rays of the same straight line, as, in this case, there are two equal shortest arcs connecting g_i and g_k. Very often $\widehat{g_ig_k}$ is taken to be the measure of the angle (g_i, g_k), and in the special case when g_i and g_k have opposite directions, we may choose $\widehat{g_ig_k} = \pi$. Another definition of the measure of an angle is obtained by considering the measure as a multi-valued function by putting $(g_i, g_k) = \widehat{g_ig_k} + 2m\pi$, m taking all integral values. Then $(g_0, g_n) = \sum (g_i, g_{i+1})$ holds, an additive value $2m\pi$ being always arbitrary. Sometimes it is useful to distinguish between an infinity of different angles bounded by g_i and g_k corresponding to the different values of m. The most useful manner to measure angles in many cases is, however, to measure them by the functions *sine* and *cosine*.

Let ϕ be a given angle bounded by two half-rays g_0, g and g_0 be taken as the positive x-axis. Also, let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, ... be points of g. The trigonometrical ratios $\cos\phi$, $\sin\phi$ are then given by

$$\frac{x_1}{|OP_1|} = \frac{x_2}{|OP_2|} = .. = \cos\phi$$

$$\frac{y_1}{|OP_1|} = \frac{y_2}{|OP_2|} = .. = \sin\phi$$



The angle ϕ is not uniquely defined by its cosine or sine alone, but by both the ratios ; and these ratios are connected by the relation $\cos^2\phi + \sin^2\phi = 1$. From the values of $\cos\phi$ and $\sin\phi$ given above, we can determine whether the ratios are positive or negative when g lies in any one of the four *quadrants* into which the axes of coordinates divide the plane. Accordingly,

$$\cos\phi = \cos(-\phi), \qquad \sin\phi = -\sin(-\phi)$$

The coordinates of the point of g at unit distance from the origin are $(\cos\phi, \sin\phi)$; and if $P = (x, y)$ is any point of g,

$$x = |OP| \cos\phi, \qquad y = |OP| \sin\phi \tag{1.3}$$

As an *application* of the last formulæ, consider a triangle OAB, and suppose that the angle between the half-rays OA and OB is ϕ. Take the positive x-axis along $\overline{OA}$ and let the coordinates of A, B be $(x_1, 0)$, (x_2, y_2). Then, as the generalisation of Pythagoras' theorem,

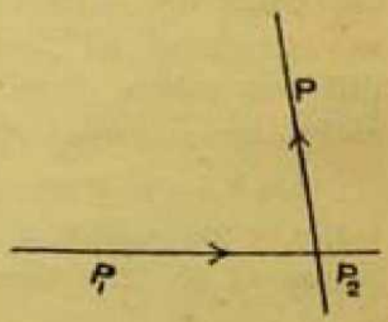
$$\begin{aligned} |AB|^2 &= (x_2-x_1)^2+y_2^2 = (x_2^2+y_2^2)+x_1^2-2x_1x_2 \\ &= |OA|^2+|OB|^2-2|OA|\ |OB| \cos\phi. \end{aligned}$$

When a vector rotates through an angle $\pi/2$, its coordinates interchange in the following manner: The coordinates (x, y) of a vector changes to $(-y, x)$ for a rotation through $+\pi/2$ and to $(y, -x)$ for a rotation through $-\pi/2$. This may be easily verified by supposing the initial point of a directed segment to coincide with the origin.

As an *application* of this rotation, suppose that two points P_1, P_2 are given and it is required to draw the perpendicular to $\overline{P_1P_2}$ at P_2 in the positive sense. Let the coordinates of P_1, P_2 be (x_1, y_1), (x_2, y_2). The coordinates and the length of $\overline{P_1P_2}$ are then

$$(x_2-x_1,\ y_2-y_1) \text{ and } |\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2\}}|$$

Rotating $\overline{P_1P_2}$ through $\pi/2$, we obtain a vector whose coordinates are $(y_1-y_2,\ x_2-x_1)$. The coordinates of the unit vector having the same direction are

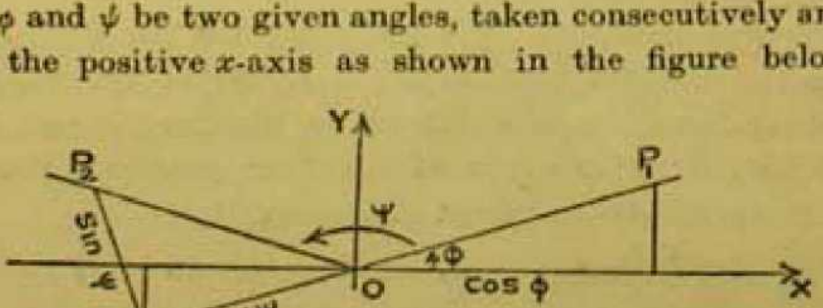


$$\left(\frac{y_1-y_2}{|P_1P_2|},\ \frac{x_2-x_1}{|P_1P_2|}\right)$$

Therefore, if P is the point of the required half-ray at unit distance from P_2, the coordinates of P are

$$\left(x_2+\frac{y_1-y_2}{|P_1P_2|},\qquad y_2+\frac{x_2-x_1}{|P_1P_2|}\right)$$

Let now ϕ and ψ be two given angles, taken consecutively and measured initially from the positive x-axis as shown in the figure below. On the



half-rays bounding these angles take points P_1, P_2 such that $|OP_1| = |OP_2|$ = unit length, and draw the perpendicular P_2P on the line OP_1.

Then $\quad P_1 = (\cos\varphi, \sin\varphi)$,

and $\quad P = (\pm \mid \cos\psi \mid \cos\varphi, \pm \mid \cos\psi \mid \sin\varphi)$,

according as $\cos\psi$ is positive or negative ; that is,

$$P = (\cos\psi\cos\varphi, \ \cos\psi\sin\varphi)$$

The coordinates of $\overline{PP_2}$ are the orthogonal projections of $\overline{PP_2}$ on the coordinate axes ; so, they are

$$(\mp \mid \sin\psi \mid \sin\varphi, \pm \mid \sin\psi \mid \cos\varphi)$$

according as $\sin\psi$ is positive or negative ; that is,

$$\overline{PP_2} = (-\sin\psi\sin\varphi, \ \sin\psi\cos\varphi)$$

Therefore $\quad P_2 = (\cos\psi\cos\varphi - \sin\psi\sin\varphi, \ \cos\psi\sin\varphi + \sin\psi\cos\varphi)$

Also, $\quad P_2 = (\cos(\varphi+\psi), \ \sin(\varphi+\psi))$

Hence, $\quad \cos(\varphi+\psi) = \cos\psi\cos\varphi - \sin\psi\sin\varphi \qquad (1.3')$

$$\sin(\varphi+\psi) = \cos\psi\sin\varphi + \sin\psi\cos\varphi$$

The formulæ (1.3′) are the *addition formulae for the cosine and the sine.* It should be observed that the proof holds for angles of all magnitudes. On replacing ψ by $-\psi$ we obtain two other formulae for $\cos(\varphi-\psi)$, $\sin(\varphi-\psi)$.

We now introduce measure of the angle between two vectors. Let two vectors α, β be defined by their coordinates (a_1, a_2), (b_1, b_2) respectively. So, their lengths are

$$|\,\alpha\,| = |\,\sqrt{(a_1^2+a_2^2)}\,|, \quad |\,\beta\,| = |\,\sqrt{(b_1^2+b_2^2)}\,|$$

Denoting by (x, α), (x, β) the angles between the positive x-axis and the vectors, we have, by (1.3),

$$\cos(x, \alpha) = \frac{a_1}{|\,\alpha\,|}, \qquad \sin(x, \alpha) = \frac{a_2}{|\,\alpha\,|}$$

$$\cos(x, \beta) = \frac{b_1}{|\,\beta\,|}, \qquad \sin(x, \beta) = \frac{b_2}{|\,\beta\,|}$$

Hence, by (1.3′),

$$\cos(\alpha, \beta) = \frac{a_1b_1 + a_2b_2}{|\,\alpha\,|\,|\,\beta\,|}, \quad \sin(\alpha, \beta) = \frac{a_1b_2 - b_1a_2}{|\,\alpha\,|\,|\,\beta\,|} \qquad (1.3'')$$

The first of these relations can be written as

$$|\,\alpha\,|\,|\,\beta\,|\cos(\alpha, \beta) = a_1b_1 + a_2b_2 \qquad (1.4)$$

The expression on the left hand side, namely, the product of the lengths of the two vectors and cosine of the angle between them, is called *the scalar product* of the two vectors. We shall denote the scalar product of two vectors α, β by the notation $\alpha . \beta$. Another notation, known as the Hamiltonian notation, is $S\alpha\beta$. The scalar product is also called the inner product or the dot product. As an application of the scalar product, the generalisation of Pythagoras' theorem can be written as

$$|AB|^2 = |CA|^2 + |CB|^2 - 2\overline{CA}.\overline{CB}$$

From the values of $\cos(\alpha, \beta)$, $\sin(\alpha, \beta)$ given above, it follows that the condition of *orthogonality* of the vectors α, β is $a_1b_1 + a_2b_2 = 0$, and the condition of *parallelism* is $a_1b_2 - b_1a_2 = 0$.

4. The straight line. Suppose we are given a straight line, a point $P_0 = (x_0, y_0)$ of it and a vector (a, b) parallel to the straight line and let $P = (x, y)$ be an arbitrary point of the straight line. Since the vector $\overline{P_0P}$ is parallel to the given vector,

$$\left.\begin{aligned} x &= x_0 + \rho a \\ y &= y_0 + \rho b \end{aligned}\right\} \quad \rho \text{ is a parameter.} \tag{1.5}$$

By giving different values to ρ we obtain different points of the straight line ; the two half-rays of the straight line emanating from P_0 are obtained by assigning to ρ positive and negative values. The equations (1.5) constitute the *parametric equations* of the straight line. Eliminating ρ, we have

$$bx - ay + (ay_0 - bx_0) = 0.$$

This is of the *general form*

$$c_1x + c_2y + c_3 = 0, \tag{1.6}$$

a linear equation in the variables. The following cases should be noticed :

(*i*) The linear equation represents a straight line unless $c_1 = c_2 = 0$. The solutions of the equation are the points of the straight line. c_1, c_2, c_3 are said to be the *coordinates* of the straight line.

(*ii*) If $c_1 = c_2 = c_3 = 0$, the equation is satisfied by any values whatever of x and y. The equation then represents the plane.

(*iii*) If $c_1 = c_2 = 0$, $c_3 \neq 0$, there is no point satisfied by the equation.

Again, let $$c_1x + c_2y + c_3 = 0$$
and $$d_1x + d_2y + d_3 = 0$$
represent the same straight line. Then

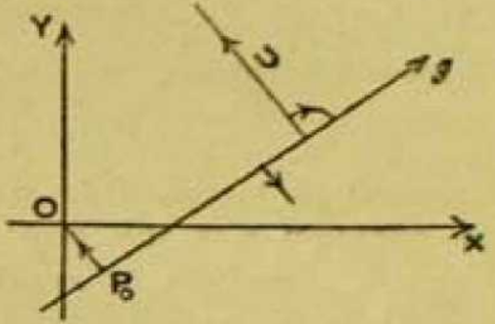
$$c_1/d_1 = c_2/d_2 = c_3/d_3 = \sigma,$$

where σ is an arbitrary quantity not equal to zero. Hence,

(*iv*) The coordinates c_1, c_2, c_3 are not uniquely determined by the straight line ; they are determined except for a common arbitrary factor other than zero.

Def. By the *slope* of a straight line $c_1x+c_2y+c_3=0$ we shall mean the ratio $-c_1/c_2$ unless $c_2=0$.

4.1. Hessian normal form and the perpendicular distance. Let u be a unit vector defined by its coordinates (a, b), so that $a^2+b^2=1$. Rotating the vector through $-\pi/2$ we obtain another vector whose coordinates are $(b, -a)$. From this latter vector we obtain, as in the last article, a straight line g, defined by



$$x=x_0+\rho b$$
$$y=y_0-\rho a$$

through a given point $P_0=(x_0, y_0)$. Eliminating ρ, we have

$$ax+by=ax_0+by_0$$

Or, putting $ax_0+by_0=-c$,

$$ax+by+c=0, \quad \text{where } a^2+b^2=1 \tag{1.7}$$

This last form (1.7) of the equation is known as the *Hessian normal form* of the equation of the straight line g.

The equations $$ax+by+c=0$$
and $$\sigma ax+\sigma by+\sigma c=0, \quad \sigma\neq 0$$

represent the same straight line. If the latter equation be also in Hessian normal form,

$$(\sigma a)^2+(\sigma b)^2=1, \quad \text{or} \quad \sigma=\pm 1$$

Therefore, there are two Hessian normal forms, differing only in sign, of the equation of the same straight line. As we obtain two half-rays of a straight line corresponding to the two signs of ρ in (1.5), we shall say that we obtain a *directed straight line* for each sign of σ. That is to say, corresponding to the two Hessian normal forms we obtain the straight line oppositely directed.

The equation of a straight line can always be reduced to a Hessian normal form. For, the equations

$$c_1x+c_2y+c_3=0$$
and $$\sigma c_1x+\sigma c_2y+\sigma c_3=0, \quad \sigma\neq 0,$$

represent the same straight line; and if the second equation be given in Hessian normal form, we must have

$$(\sigma c_1)^2+(\sigma c_2)^2=1, \quad \text{or} \quad |\sigma|=1/|\sqrt{(c_1^2+c_2^2)}|$$

In this case $(\sigma c_1, \sigma c_2)$ is a unit vector perpendicular to the straight line and therefore (c_1, c_2) is a vector of length $|\sqrt{(c_1^2+c_2^2)}|$ perpendicular to the same straight line.

Let the equation of a straight line g in Hessian normal form be $ax+by+c=0$. From the last article we have

$$c=-(ax_0+by_0)=-(u.\,\overline{OP_0})=u.\,\overline{P_0O},$$

$$=\text{the scalar product of } u \text{ and } \overline{P_0O},$$

where u is the unit vector (a, b). If the vector u and $\overline{P_0O}$ are parallel,

$$u.\,\overline{P_0O}=\pm\,|\,u\,|\,|\,P_0O\,|=\pm\,|\,P_0O\,|$$

according as u and $\overline{P_0O}$ have the same or opposite directions. Hence, the quantity c is the *perpendicular distance* of the origin from g; this distance is positive or negative according as u and $\overline{P_0O}$ have the same or opposite directions.

Again, let (ξ, η) be the coordinates of a given point and (x', y') those of the foot of the perpendicular drawn from this point to the given straight line g. The vectors $(\xi-x', \eta-y')$, and (a, b) are then parallel; and so their scalar product gives the perpendicular distance of the given point from the given straight line. Denoting this distance by d,

$$d=a(\xi-x')+b(\eta-y')=a\xi+b\eta+c$$

Hence we are led to the following conclusion :

If $ax+by+c=0$ is the equation of a given straight line in Hessian normal form and (ξ, η) the coordinates of a given point, then the quantity $a\xi+b\eta+c$ gives the perpendicular distance of the given point from the given straight line ; this distance is positive or negative according as the vectors (a, b) and $(\xi-x', \eta-y')$, where (x', y') are the coordinates of the foot of the perpendicular, have the same or the opposite directions.

5. Straight line and triangle. Let it be required to find the equation of the straight line joining two distinct points (x_1, y_1) and (x_2, y_2). Suppose the equation is

$$c_1x+c_2y+c_3=0$$

Then

$$c_1x_1+c_2y_1+c_3=0, \quad \text{and} \quad c_1x_2+c_2y_2+c_3=0$$

These three equations form a system of linear homogeneous equations in the unknowns c_1, c_2, c_3. The necessary and sufficient condition that a solution other than (0, 0, 0) exists is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0 \tag{1.8}$$

This equation can be written in the form (1.6) as

$$c_1 x + c_2 y + c_3 = 0$$

where $\qquad c_1 = y_1 - y_2, \quad c_2 = x_2 - x_1, \quad c_3 = x_1 y_2 - y_1 x_2$

As the given points are supposed to be distinct, c_1 and c_2 cannot vanish simultaneously, and therefore (1.8) is the equation of the straight line.

The rank of the matrix whose determinant is the left-hand side of (1.8) is equal to two, as at least one of the three second-order determinants c_1, c_2, c_3 is other than zero. Hence, the solution (c_1, c_2, c_3) is determined except for an arbitrary common factor. This algebraic result corresponds to the geometrical fact that there exists one and only one straight line joining two distinct points.

The equation (1.8) can also be interpreted in another way. As the determinant on the left-hand side is equal to zero, the three rows are dependent. The second and the third rows are obviously independent and so the first row is dependent on them. This means that there exists two numbers γ and λ satisfying the relations

$$\begin{aligned} x &= \gamma x_1 + \lambda x_2 \\ & \qquad\qquad \gamma + \lambda = 1 \\ y &= \gamma y_1 + \lambda y_2 \end{aligned} \tag{1.9}$$

We shall derive these equations again in the next chapter. A third interpretation of (1.8) is given below.

The *area* of a triangle is defined as half the product of the lengths of any two sides and the sine of the angle included between them. Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$ be the vertices of the triangle and let the vectors $\overline{P_3P_1}$, $\overline{P_3P_2}$ be denoted by α, β. The area is then given by

$$\Delta = \tfrac{1}{2} \mid \alpha \mid \mid \beta \mid \sin(\alpha, \beta),$$

and is considered positive or negative according as the sense in which the angle is measured is positive or negative. The coordinates of α and β are

$$(x_1 - x_3, y_1 - y_3) \quad \text{and} \quad (x_2 - x_3, y_2 - y_3).$$

By (1.3″)

$$\sin(\alpha, \beta) = \frac{(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)}{\mid \alpha \mid \mid \beta \mid}$$

Therefore

$$\Delta = \tfrac{1}{2} \begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} = \tfrac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{1.10}$$

From this result we notice that

(*i*) The area remains invariant for cyclical permutations of the vertices, but changes sign for other permutations. Thus, denoting the above area by $\Delta(P_1P_2P_3)$,

$$\Delta(P_1P_2P_3)=\Delta(P_2P_3P_1)=-\Delta(P_2P_1P_3)$$

(*ii*) The area vanishes by (1.8) if the vertices lie on a straight line. We may accordingly interpret the equation (1.8) of a straight line as the vanishing of the area of a triangle formed by three points of the straight line.

(*iii*)
$$\Delta(P_1P_2P_3)=\tfrac{1}{2}\begin{vmatrix} x_1-x_0 & y_1-y_0 & 1\\ x_2-x_0 & y_2-y_0 & 1\\ x_3-x_0 & y_3-y_0 & 1\end{vmatrix}$$

$$=\tfrac{1}{2}\left[\begin{vmatrix} x_2-x_0 & y_2-y_0\\ x_3-x_0 & y_3-y_0\end{vmatrix}+\begin{vmatrix} x_3-x_0 & y_3-y_0\\ x_1-x_0 & y_1-y_0\end{vmatrix}+\begin{vmatrix} x_1-x_0 & y_1-y_0\\ x_2-x_0 & y_2-y_0\end{vmatrix}\right]$$

$$=\Delta(P_2P_3P_0)+\Delta(P_3P_1P_0)+\Delta(P_1P_2P_0),$$

where $P_0=(x_0. y_0)$ is any arbitrary point in the plane. This is the *addition formula* for areas of triangles.

Note. We have $2\Delta=|\alpha||\beta|\sin(\alpha,\beta)$. So, 2Δ is a function of the two vectors α, β. This function possesses the following properties :

(*a*) If we multiply one of the vectors by any quantity, the function is multiplied by the same quantity :

Thus, $|c\alpha||\beta|\sin(c\alpha,\beta)=c|\alpha||\beta|\sin(\alpha,\beta)$;
for, $|c\alpha|=|c||\alpha|$ and $\sin(c\alpha,\beta)=\pm\sin(\alpha,\beta)$

according as c is positive or negative.

(*b*) The function remains unchanged if we replace one of the vectors by the sum of the two :

Thus, $|\alpha+\beta||\beta|\sin(\alpha+\beta,\beta)=|\alpha||\beta|\sin(\alpha,\beta)$.

This can be verified directly by expressing the two sides in terms of the coordinates of the vectors. This property implies that two triangles which have the same base and the same altitude have the same area.

(*c*) If $\epsilon_1=(1, 0)$, $\epsilon_2=(0, 1)$ are two unit vectors, the same function of ϵ_1 and ϵ_2 as 2Δ is of α and β has the value unity :

$$|\epsilon_1||\epsilon_2|\sin(\epsilon_1,\epsilon_2)=1.$$

On account of these three properties the function 2Δ is the determinant given above.*

---

* See F. W. Levi, Algebra, Chapter I.

CHAPTER II

CROSS-RATIO

6. Cross-ratio of four collinear points. The totality of points lying on a straight line is said to form a *row* (*or range*) *of points*, or simply, a row. The straight line is called the *base* of the row.

Let $P_1=(x_1, y_1)$, $P_2=(x_2, y_2)$ be two given points. If $P=(x, y)$ is an arbitrary point of the straight line joining P_1 and P_2, then

$$\overline{P_1P}=\lambda\,\overline{P_1P_2}, \text{ where } \lambda \text{ is an arbitrary constant.}$$

In coordinates, $\quad x-x_1=\lambda(x_2-x_1),\ \ y-y_1=\lambda(y_2-y_1)$

Or, putting $\quad \gamma=1-\lambda,$

$$x=\gamma x_1+\lambda x_2,\ \ y=\gamma y_1+\lambda y_2,\ \ 1=\gamma+\lambda$$

These equations constitute a representation of the row whose base is the straight line P_1P_2. The equations are exactly the same as the equations (1.9) ; but we have now a geometrical interpretation of γ and λ. We have

$$\lambda=\overline{P_1P}/\overline{P_1P_2}$$

Therefore $\quad \gamma=1-\lambda=\overline{P_1P_2}/\overline{P_1P_2}-\overline{P_1P}/\overline{P_1P_2}$

$$=(\overline{P_1P_2}+\overline{PP_1})/\overline{P_1P_2}=\overline{PP_2}/\overline{P_1P_2}$$

Hence $\quad -\lambda/\gamma=\overline{P_1P}/\overline{P_2P}=\overline{PP_1}/\overline{PP_2}$

As this is a ratio of (algebraic) distances of a point from two distinct points, it cannot be equal to unity. Again, let this ratio be given by $-\nu/\mu$.

Then $\quad -\nu\neq\mu,\ \text{ or }\ \mu+\nu\neq 0$

So, we may write $\quad \nu/\mu=\dfrac{\nu}{\mu+\nu}\Big/\dfrac{\mu}{\mu+\nu}$

Therefore, we may put $\quad \gamma=\dfrac{\mu}{\mu+\nu},\ \ \lambda=\dfrac{\nu}{\mu+\nu}$

Hence, a representation of the row is given by

$$x=\frac{\mu}{\mu+\nu}x_1+\frac{\nu}{\mu+\nu}x_2, \qquad y=\frac{\mu}{\mu+\nu}y_1+\frac{\nu}{\mu+\nu}y_2, \qquad \mu+\nu\neq 0$$

The reason why we take two parameters γ, λ, of which one is dependent and uniquely determined by the other, will become obvious later on. Sometimes, it is useful to use the representation

$$\overline{P_1P} = \lambda \overline{P_1P_2},$$

where only one parameter occurs. For example, we can interpret this equation in a cinematic sense, λ being the time. The equation then means that the point P is moving on the line with constant velocity.

Further, let P' be another point of the straight line corresponding to the constants (γ', λ'). Then

$$-\lambda'/\gamma' = \overline{P'P_1}/\overline{P'P_2}$$

Therefore
$$\frac{\overline{PP_1}}{\overline{PP_2}} \bigg/ \frac{\overline{P'P_1}}{\overline{P'P_2}} = \frac{\overline{PP_1} \cdot \overline{P'P_2}}{\overline{PP_2} \cdot \overline{P'P_1}} = \frac{\lambda\gamma'}{\gamma\lambda'}$$

The left-hand side expression, which is a ratio of the distance-ratios, is called a *cross-ratio of the four collinear points*. We shall denote this cross-ratio by the notation (P_1P_2, PP'), and so write

$$(P_1P_2, PP') = \lambda\gamma'/\gamma\lambda' \tag{2.1}$$

7. Cross-ratio of four concurrent lines. The totality of straight lines in the plane passing through a point is said to form a *pencil of lines*. The straight lines are called the *rays* and the common point the *centre* of the pencil. All straight lines parallel to one another are said to form a pencil of parallel lines.

Let the equations of two intersecting straight lines p_1 and p_2 be $l_1(x, y) = 0$ and $l_2(x, y) = 0$, where l_1 and l_2 are linear functions of the variables. Then the equation

$$\gamma l_1 + \lambda l_2 = 0, \tag{2.2}$$

where γ and λ are two arbitrary constants other than both zero, represents a straight line p passing through the point of intersection of p_1 and p_2. For, since p_1 and p_2 are supposed to be non-parallel, the equation is a linear equation in which the coefficients of both x and y cannot vanish simultaneously; moreover, the coordinates of the point which make both l_1 and l_2 zero also make $\gamma l_1 + \lambda l_2$ zero. On the other hand, every straight line p which passes through the point of intersection of p_1 and p_2 has its equation of the form (2.2), where γ and λ are two arbitrary constants other than both zero. For, let $l_1 \equiv a_1x + b_1y + c_1$, $l_2 \equiv a_2x + b_2y + c_2$ and (x_0, y_0) the common point; also let $l \equiv ax + by + c = 0$ be an arbitrary straight line passing through (x_0, y_0). Then, since the three equations

$$a_1x_0+b_1y_0+c_1=0$$
$$a_2x_0+b_2y_0+c_2=0$$
$$a\,x_0+by_0\;+c\;=0$$

hold simultaneously, we must have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a & b & c \end{vmatrix}=0$$

This shows that there exist two numbers γ, λ, not both zero, such that

$$a=\gamma a_1+\lambda a_2,\quad b=\gamma b_1+\lambda b_2,\quad c=\gamma c_1+\lambda c_2$$

Therefore, the equation $l=0$ can be written in the form (2.2). Accordingly, in view of the arbitrariness of the constants γ, λ, the equations (2.2) represent a pencil of lines. Also, for $\rho \neq 0$, the pairs (γ, λ) and $(\rho\gamma, \rho\lambda)$ obviously represent the same ray of the pencil.

For the sake of simplicity, suppose that the equations $l_1=0$, $l_2=0$ are given in Hessian normal forms (1.7), and a point $P=(x, y)$ be taken on p. Draw perpendiculars PQ_1, PQ_2 on p_1, p_2 respectively. By § 4.1

$$\overline{Q_1P}/\overline{Q_2P}=\overline{PQ}_1/\overline{PQ}_2=-\lambda/\gamma$$

Therefore
$$\frac{\sin(p, p_1)}{\sin(p, p_2)}=\frac{\sin(p_1, p)}{\sin(p_2, p)}=-\frac{\lambda}{\gamma}$$

The centre of the pencil divides each ray into two half-rays. The angles (p, p_1) and (p, p_2) are measured between that half-ray of p on which P lies and those half-rays of p_1 and p_2 on which Q_1 and Q_2 lie, in the directions of $\overline{PQ}_1$ and $\overline{PQ}_2$ respectively. Take another ray p' corresponding to the constants (γ', λ'). Then

$$\frac{\sin(p', p_1)}{\sin(p', p_2)}=-\frac{\lambda'}{\gamma'}$$

Therefore

$$\frac{\sin(p, p_1)}{\sin(p, p_2)}\Big/\frac{\sin(p', p_1)}{\sin(p', p_2)}=\frac{\sin(p, p_1)\sin(p', p_2)}{\sin(p, p_2)\sin(p', p_1)}=\frac{\lambda\gamma'}{\gamma\lambda'}$$

The expression on the left-hand side is called a *cross-ratio of the four concurrent straight lines*. We shall denote this cross-ratio by the notation (p_1p_2, pp').

It may be seen from the expression of the cross-ratio that the cross-ratio remains the same if the equations of p_1 and p_2 are given in forms other than in Hessian normal forms. For, let the equations $l_1=0$, $l_2=0$ of p_1, p_2 be given in general forms. These equations can evidently be transformed in Hessian normal forms by multiplying by proper constants.

Put
$$L_1 = l_1/\rho_1, \quad L_2 = l_2/\rho_2,$$
$$L = (\gamma l_1 + \lambda l_2)/\rho, \quad L' = (\gamma' l_1 + \lambda' l_2)/\rho',$$

the constants ρ_1, ρ_2 being so chosen that $L_1 = 0$, $L_2 = 0$ are in Hessian normal forms. So

$$L = \gamma\frac{\rho_1}{\rho}L_1 + \lambda\frac{\rho_2}{\rho}L_2 = \Gamma L_1 + \Lambda L_2, \text{ say} \qquad \text{where } \Gamma = \gamma\frac{\rho_1}{\rho}, \quad \Lambda = \lambda\frac{\rho_2}{\rho}$$

Similarly $\quad L' = \Gamma' L_1 + \Lambda' L_2, \quad$ where $\quad \Gamma' = \gamma'\frac{\rho_1}{\rho'}, \quad \Lambda' = \lambda'\frac{\rho_2}{\rho'}$.

Hence, the cross-ratio

$$(p_1p_2, pp') = \frac{\Lambda\Gamma'}{\Gamma\Lambda'} = \frac{\lambda\gamma'}{\gamma\lambda'}$$

Thus, the cross-ratio is independent of the constants $\rho_1, \rho_2, \rho, \rho'$, *i.e.*, independent of the forms of the equations $l_1 = 0$, $l_2 = 0$. The cross-ratio is equal to unity if and only if the angles $(p, p_1) = (p', p_1)$, *i.e.*, if the lines p and p' coincide. If therefore $\gamma/\lambda \neq \gamma'/\lambda'$, the lines p, p' are different.

7.1. Cross-ratio of projection and section. Let us start with a row of points on a base p_0. Take a point P_0 external to p_0 and join P_0 with the points of the row so as to obtain straight lines through P_0. We are then said *to project* the row from P_0. If P is a point of the row, the straight line P_0P is called the projection of P from P_0. All these projections, together with the parallel to p_0 through P_0, form the rays of a pencil of lines. On the other hand, we may start with a pencil of lines and take a straight line p_0, not passing through the centre of the pencil, to cut the rays of the pencil in points forming a row. The row is then called the *section* of the pencil by p_0. If p is a ray of the pencil, the point of intersection of p and p_0 is called the section of p by p_0.

✓ Let a straight line be given as the join of two points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and let $P_0 = (x_0, y_0)$ be an external point. The equation of the straight line joining P_0 and a point (ξ, η) is

$$\begin{vmatrix} x - x_0 & y - y_0 \\ \xi - x_0 & \eta - y_0 \end{vmatrix} = 0$$

If (ξ, η) is a point of the straight line P_1P_2, the equation reduces, by § 6, to

$$\begin{vmatrix} x - x_0 & y - y_0 \\ \gamma x_1 + \lambda x_2 - x_0 & \gamma y_1 + \lambda y_2 - y_0 \end{vmatrix} = 0,$$

where $\gamma + \lambda = 1$. This equation can be written as

$$\gamma l_1 + \lambda l_2 = 0,$$

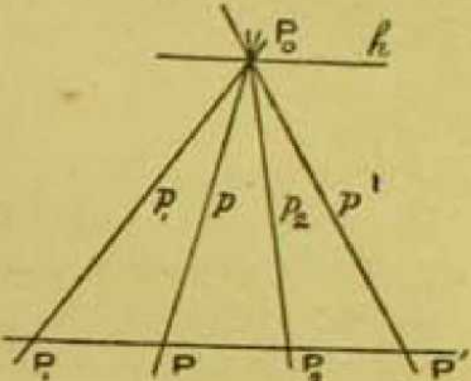
$l_1=0$, $l_2=0$ being, as before, the equations of the straight lines p_1 and p_2 joining P_0 to P_1 and P_2 respectively. To every pair (γ, λ) satisfying $\gamma+\lambda=1$, there corresponds exactly one projection from P_0.

Put $$\gamma=\frac{\mu}{\mu+\nu}, \quad \lambda=\frac{\nu}{\mu+\nu}, \qquad \text{where } \mu+\nu \neq 0$$

Then $$\mu l_1+\nu l_2=(\mu+\nu)\,(\gamma l_1+\lambda l_2)$$

Therefore $\mu l\ +\nu l_2=0$ and $\gamma l_1+\lambda l_2=0$ define the same projection. Hence $\mu+\nu\neq 0$ is a sufficient condition that the line $\mu l_1+\nu l_2=0$ is the projection of a point of the line P_1P_2 from P_0.

It may be noticed that if we choose μ, ν such that $\mu+\nu=0$, we obtain the equation $l_1-l_2=0$ representing the parallel h to P_1P_2 through P_0.



Let the coordinates of two points P, P' of the straight line P_1P_2 be

$$(\gamma x_1+\lambda x_2,\ \gamma y_1+\lambda y_2), \quad (\gamma' x_1+\lambda' x_2,\ \gamma' y_1+\lambda' y_2),$$

where $$\gamma+\lambda=\gamma'+\lambda'=1$$

The equations of the straight lines p, p' joining P_0 to P, P' are then

$$\gamma l_1+\lambda l_2=0, \quad \gamma' l_1+\lambda' l_2=0$$

Therefore, by §§ 6, 7, the cross-ratios have the following value :

$$(P_1P_2,\ PP')=(p_1p_2,\ pp')=\lambda\gamma'/\gamma\lambda'$$

The quantity on the right-hand side is evidently independent of the position of the point P_0. Accordingly, the cross-ratio is unaltered by projection from any external point. Similarly, the cross-ratio is unaltered by section by any transversal. We thus arrive at the following conclusion : *The cross-ratio is unaltered by projection or section.*

Since the equation of the parallel h to P_1P_2 is $l_1-l_2=0$, the cross-ratio

$$(p_1p_2,\ ph)=-\lambda/\gamma$$

Thus, if the first three of four concurrent straight lines p_1, p_2, p, p', be cut by a transversal parallel to the fourth in points P_1, P_2, P, then

$$(p_1p_2,\ pp')=\overline{PP_1}/\overline{PP_2}$$

Now, suppose that we are given three distinct points

$$P_1=(x_1, y_1),\quad P_2=(x_2, y_2),\quad P=(\gamma x_1+\lambda x_2, \gamma y_1+\lambda y_2),\quad \gamma+\lambda=1$$

and a ratio a/b. Is there a point P' collinear with the three such that $(P_1P_2, PP')=a/b$?

Let $$P'=(\gamma' x_1+\lambda' x_2,\quad \gamma' y_1+\lambda' y_2),\quad \gamma'+\lambda'=1$$

Then $$a/b=\lambda\gamma'/\gamma\lambda'$$

Therefore $$a=\rho\lambda\gamma',\ b=\rho\gamma\lambda',\ \rho\neq 0,$$

or $$\gamma'=a/\rho\lambda,\quad \lambda'=b/\rho\gamma,\quad \text{where}\quad 1=\gamma'+\lambda'=(a/\lambda+b/\gamma)/\rho$$

Or $$\rho=a/\lambda+b/\gamma$$

Therefore, γ' and λ' can be determined unless $a/\lambda+b/\gamma=0$. Thus, we can determine the point P' unless $a/b=-\lambda/\gamma$.

On the other hand, when three rays p_1, p_2, p of a pencil and the value of the cross-ratio (p_1p_2, pp') are given, the ray p' can be determined uniquely.

8. The six cross-ratios of the twenty-four permutations. Let x_1, x_2, x, x' be the coordinate distances of four collinear points P_1, P_2, P, P' from any chosen origin on the straight line, as described in § 1. Then

$$(P_1P_2, PP')=\frac{x_1-x}{x_2-x}\Big/\frac{x_1-x'}{x_2-x'}=\frac{(x_1-x)\,(x_2-x')}{(x_2-x)\,(x_1-x')}.$$

As there are twenty-four permutations of four different elements, so we obtain twenty-four cross-ratios from these four points. But these twenty-four cross-ratios are not all different. From the value of the cross-ratio given above, it may be easily verified that

$$(P_1P_2, PP')=(P_2P_1, P'P)=(PP', P_1P_2)=(P'P, P_2P_1)$$
$$(P_1P_2, P'P)=(P_2P_1, PP')=(P'P, P_1P_2)=(PP', P_2P_1)$$
$$(P_1P, P_2P')=(PP_1, P'P_2)=(P_2P', P_1P)=(P'P_2, PP_1)$$
$$(P_1P, P'P_2)=(PP_1, P_2P')=(P'P_2, P_1P)=(P_2P', PP_1)$$
$$(P_1P', P_2P)=(P'P_1, PP_2)=(P_2P, P_1P')=(PP_2, P'P_1)$$
$$(P_1P', PP_2)=(P'P_1, P_2P)=(PP_2, P_1P')=(P_2P, P'P_1)$$

Therefore, *only six of the twenty-four cross-ratios may be distinct.* It is seen from above that the two pairs of letters separated by a comma, occurring in our notation, are such that we can, without altering the cross-ratio, interchange the letters of the first pair and of the second pair simultaneously, or interchange the two pairs of letters.

BCU 2691

Further, if we take the first six cross-ratios on the left-hand side of the above six sets of relations, it may be verified that

$$(P_1P_2, PP')\,(P_1P_2, P'P)=1,$$

$$(P_1P, P_2P')\,(P_1P, P'P_2)=1, \qquad (P_1P', P_2P)\,(P_1P', PP_2)=1$$

and

$$(P_1P_2, PP')+(P_1P, P_2P')=1,$$

$$(P_1P_2, P'P)+(P_1P', P_2P)=1, \qquad (P_1P, P'P_2)+(P_1P', PP_2)=1$$

Owing to the existence of these two sets of relations, the six distinct cross-ratios are not all independent. If any one of them is given, the remaining five can be determined as functions of the given one.

Thus, if $(P_1P_2, PP')=\delta$,

$$(P_1P_2, P'P)=1/\delta, \quad (P_1P, P_2P')=1-\delta, \quad (P_1P', P_2P)=(\delta-1)/\delta,$$

$$(P_1P, P'P_2)=1/(1-\delta), \qquad (P_1P', PP_2)=\delta/(\delta-1).$$

All we have said above regarding the six cross-ratios of four collinear points apply equally well about the cross-ratios of four concurrent straight lines.

8.1. Special cases. If we suppose that (1) any two of the four points are ultimately coincident, or (2) any two of the six cross-ratios have the same value, we obtain special cases where the six values are not all distinct.

(1) Let the points P_1, P_2, P be distinct.

If P' coincides with P, then ultimately

$$\delta=1/\delta=1/1, \quad 1-\delta=(\delta-1)/\delta=0/1, \quad 1/(1-\delta)=\delta/(\delta-1)=1/0$$

If P' coincides with P_2, then ultimately

$$1-\delta=1/(1-\delta)=1/1, \quad \delta=\delta/\ \delta-1)=0/1, \quad 1/\delta=(\delta-1)/\delta=1/0$$

And we obtain similar results when P' coincides with P_1. Therefore, we have the three values, 1, 0, 1/0, each repeated twice; the last value is, of course, meaningless without the notion of limit.

(2) (*i*) Let $\delta=1/\delta$. So, $\delta=\pm 1$. Taking $\delta=-1$,

$$\delta=1/\delta=-1, \quad 1-\delta=(\delta-1)/\delta=2, \quad 1/(1-\delta)=\delta/(\delta-1)=1/2$$

Therefore, we have the values -1, 2, 1/2, each repeated twice. The case $\delta=+1$ has been considered in (1) above.

(*ii*) Let $\delta=1-\delta$. So, $\delta=1-\delta=1/2$,

$$1/\delta=1/(1-\delta)=2, \qquad (\delta-1)/\delta=\delta/(\delta-1)=-1$$

We obtain the same three values as in (*i*), but not for the same cross-ratios.

(*iii*) Let $\delta=\delta/(\delta-1)$. So, $\delta=0, 2$. Taking $\delta=2$,

$$\delta=\delta/(\delta-1)=2, \quad 1/\delta=(\delta-1)/\delta=1/2, \quad 1-\delta=1/(1-\delta)=-1$$

22859

We have here similar result as in (*ii*). The case $\delta=0$ has been considered in (1) above.

(*iv*) Let $\delta=1/(1-\delta)$. So, $\delta^2-\delta+1=0$.

This is an equation of the second degree with imaginary roots. Each root is repeated thrice in the six cross-ratios.

(*v*) Let $\delta=(\delta-1)/\delta$. So, $\delta^2-\delta+1=0$.

We have here similar result as in (*iv*).

9. Harmonic division. Let A, B, C, D be four collinear points such that

$$(AB, CD)=-1 \tag{2.1'}$$

Then

$$\frac{\overline{CA}}{\overline{CB}}\Big/\frac{\overline{DA}}{\overline{DB}}=-1, \quad \text{or} \quad \frac{\overline{AC}}{\overline{BC}}+\frac{\overline{AD}}{\overline{BD}}=0,$$

or

$$\frac{\overline{AC}}{\overline{AC}-\overline{AB}}+\frac{\overline{AD}}{\overline{AD}-\overline{AB}}=0$$

Or

$$2\overline{AC}.\,\overline{AD}=\overline{AB}.\,(\overline{AC}+\overline{AD})$$

Hence, the segments $\overline{AC}$, $\overline{AB}$, $\overline{AD}$ are in harmonic progression. In this case, we call the four points A, B, C, D *the four harmonic points*. We say that the segment AB is harmonically divided by the segment CD, or the points A, B (C, D) are *harmonically separated* by the points C, D (A, B). The two points A, B (C, D) are said to be *harmonic conjugates* of one another with respect to the two points C, D (A, B).

When $(AB, CD)=-1$,

$$\begin{aligned}&(AB, CD)=(BA, CD)=(AB, DC)=(BA, DC)\\ &=(CD, AB)=(DC, AB)=(CD, BA)=(DC, BA)\end{aligned}$$

Thus, there are eight cross-ratios which are harmonic. When the points are harmonic, the two letters A, B or the two letters C, D or the pairs of letters (AB), (CD) or the letters and the pairs can be interchanged in our notation without altering the cross-ratio.

Referring to the special cases considered in the last article, we find that in (*i*) the two points P_1, P_2 are harmonically separated by the two points P, P', in (*ii*) P_1, P' separate P_2, P harmonically and in (*iii*) P_1, P separate P_2, P' harmonically.

Further, it is evident that if C lies within the segment AB, D must lie outside, and *vice versa*. Also, if we suppose that A, B are fixed while C, D are variable, it may be verified from the formula (2.3) below that both C, D approach A (or B) simultaneously, and when C is the middle point of the segment AB, D cannot be located.

The relation $(AB, CD) = -1$ can be put into an useful form. Let O be the middle point of the segment AB.

Then $$\frac{\overline{OC}-\overline{OA}}{\overline{OC}+\overline{OA}}=\frac{\overline{OA}-\overline{OD}}{\overline{OA}+\overline{OD}}, \quad \text{or} \quad \frac{\overline{OC}}{\overline{OA}}=\frac{\overline{OA}}{\overline{OD}}$$

Therefore $$|OA|^2=\overline{OC}.\overline{OD} \tag{2.3}$$

The converse is also true, namely that if (2.3) holds, then $(AB, CD) = -1$.

Four harmonic lines are defined in the same way as four harmonic points. Thus, if four concurrent straight lines a, b, c, d are such that $(ab, cd) = -1$, then a, b, c, d are four harmonic lines in which a, b (c, d) are harmonically separated by c, d (a, b). Let the equations of a, b, c, d be

$$l_1=0,\ l_2=0,\ \gamma l_1+\lambda l_2=0,\ \gamma' l_1+\lambda' l_2=0$$

respectively, where l_1 and l_2 are linear functions of the variables and $\gamma, \lambda, \gamma', \lambda'$ constants.

Let $$\lambda\gamma'/\gamma\lambda'=-1 \quad \text{and} \quad L_1=\gamma l_1,\ L_2=\lambda l_2.$$

Then the equations of the four harmonic lines are expressed in the normalised forms :

$$L_1=0,\ L_2=0,\ L_1+L_2=0,\ L_1-L_2=0.$$

If, in particular, the equations $L_1=0$, $L_2=0$ are given in Hessian normal forms, then $L_1+L_2=0$, $L_1-L_2=0$ are the external and internal bisectors of the angle between $L_1=0$, $L_2=0$ and are therefore perpendicular to one another. Thus, *two intersecting straight lines and the internal and external bisectors of the angle between them form four harmonic lines* ; or, if the two arms of a right angle are separated harmonically by two straight lines, then the arms are the internal and external bisectors of the angles between the straight lines.

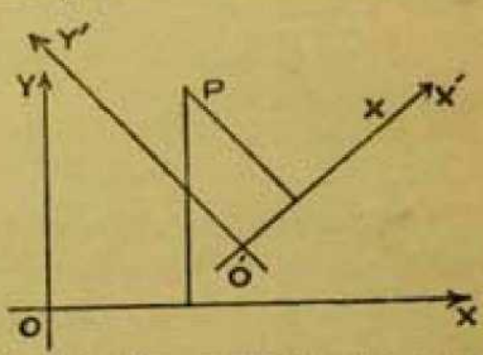
Finally, since the cross-ratio is unaltered by projection or section, it follows that projections and sections of four harmonic elements are four harmonic elements.

CHAPTER III

RIGID MOTIONS

10. Change of coordinate axes. The consideration of different kinds of transformations of coordinates is a fundamental aspect of geometrical study. One of the important purposes of such consideration is the classification of geometrical properties. We shall consider here the transformation of coordinates from one system of orthogonal axes to another, both being right-handed.

Let us take a straight line and let its equation in Hessian normal form be $ax+by+c_1=0$. Take this straight line as the new y-axis and call it the y'-axis. We have noticed in § 4.1 that the perpendicular distance of any point (x, y) from this straight line is $ax+by+c_1$ and is positive or negative according as the direction of the perpendicular and of the vector (a, b) are the same or the opposite. So, take the positive direction of the new x-axis, called the x'-axis, in the direction of the vector (a, b). Let this vector be rotated through $\pi/2$ so as to give the positive direction along the y'-axis in the direction of the vector $(-b, a)$.



Thus, if a point P has the coordinates (x, y) and (x', y') with reference to the old and new axes of coordinates respectively, then the transformation of coordinates from the old to new system is given by

$$\begin{aligned} x' &= ax+by+c_1 \\ y' &= -bx+ay+c_2 \end{aligned} \qquad a^2+b^2=1. \tag{3.1}$$

The variables x', y' are linear functions of the variables x, y. The equations of the x'- and y'- axes are

$$-bx+ay+c_2=0$$

and

$$ax+by+c_1=0$$

respectively and the new origin O' is the intersection of these axes.

We have (§ 3)

$$a=\cos(x, x')=\cos(y, y')$$

$$b=\sin(x, x')=\cos(y, x')=\cos(-x, y')=-\cos(x, y')$$

Hence, the transformation (3. 1) can be written in various forms, two of which are

$$x' = \cos(x, x')\,x + \sin(x, x')\,y + c_1,$$
$$y' = -\sin(x, x')\,x + \cos(x, x')\,y + c_2$$

and

$$x' = \cos(x, x')\,x + \cos(y, x')\,y + c_1$$
$$y' = \cos(x, y')\,x + \cos(y, y')\,y + c_2$$

Further, as the determinant of the coefficients

$$\begin{vmatrix} a & b \\ -b & a \end{vmatrix} = a^2 + b^2 = 1$$

is other than zero, we can solve the equations (3. 1) and obtain

$$\left.\begin{aligned} x &= ax' - by' - ac_1 + bc_2 \\ y &= bx' + ay' - bc_1 - ac_2 \end{aligned}\right. \qquad a^2 + b^2 = 1 \tag{3.2}$$

The transformation (3.2) is called the *inverse* of (3.1).

10.1 Invariants of the transformation. We may look upon equations (3.1) and the inverse (3.2) from another point of view. Instead of regarding them as representing a change in the system of coordinate axes, we may interpret them as a *one-to-one correspondence between the points of the plane* satisfying certain condition. Thus, a point $P = (x, y)$ is carried into a point $P' = (x', y')$ by means of (3.1) and P' is carried back to P by means of (3.2).

Since the transformation we are considering is linear, it is evident that the degree of a polynomial is unaltered by the transformation. So, in particular, a straight line is transformed into a straight line. Also, a vector is transformed into a vector. Suppose that a vector $\alpha = (a_1, a_2)$ is transformed by (3.1) into a vector $\alpha' = (a'_1, a'_2)$ and suppose, without loss of generality, that

$$a_1 = x_2 - x_1, \qquad a_2 = y_2 - y_1$$
$$a'_1 = x'_2 - x'_1, \qquad a'_2 = y'_2 - y'_1$$

where (x_1, y_1), (x_2, y_2) are transformed into (x'_1, y'_1), (x'_2, y'_2). Then

$$\left.\begin{aligned} a'_1 &= aa_1 + ba_2 \\ a'_2 &= -ba_1 + aa_2 \end{aligned}\right. \qquad a^2 + b^2 = 1 \tag{3.3}$$

The transformation of the coordinates of a vector is thus independent of c_1, c_2. Further, the length of a vector and scalar product of two vectors remain unaltered. For, supposing that (b_1, b_2), (b'_1, b'_2) are the coordinates of a vector β and of its transform β', we have by (3.3),

$$\alpha'.\beta' = a'_1b'_1 + a'_2b'_2 = a_1b_1 + a_2b_2 = \alpha.\beta$$

And for $\beta = \alpha$, we must have $\beta' = \alpha'$.

So $$|\alpha'|^2 = \alpha'.\alpha' = \alpha.\alpha = |\alpha|^2$$

It follows that the cosine of the angle between two vectors remains unaltered. The sine of the angle also remains unaltered. For,

$$|\alpha'|\,|\beta'|\sin(\alpha', \beta') = \begin{vmatrix} a'_1 & a'_2 \\ b'_1 & b'_2 \end{vmatrix} = \begin{vmatrix} aa_1 + ba_2 & -ba_1 + aa_2 \\ ab_1 + bb_2 & -bb_1 + ab_2 \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ -b & a \end{vmatrix} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = |\alpha|\,|\beta|\sin(\alpha, \beta)$$

Therefore $$\sin(\alpha', \beta') = \sin(\alpha, \beta)$$

Thus, the angle between two vectors remains unaltered.

Hence, *the distance, the angle and therefore the area remain invariant.* It is on account of this property that the transformation (3.1) is called *a rigid motion*, or a rigid displacement of the plane. The inverse of a rigid motion is a rigid motion.

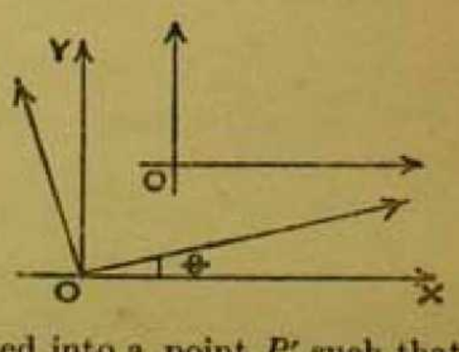
11. Translation and rotation. If in equations (3.1) we put $a = 1$, we obtain

$$\begin{aligned} x' &= x + c_1 \\ y' &= y + c_2 \end{aligned} \qquad (3.4)$$

This transformation is called a *translation* or a *parallel displacement*. And if we put $c_1 = c_2 = 0$ in (3.1), we obtain

$$\begin{aligned} x' &= ax + by \\ y' &= -bx + ay \end{aligned} \qquad a^2 + b^2 = 1 \qquad (3.5)$$

This transformation is called a *rotation* about the origin O. Translations and rotations are special cases of rigid motions.



In a rigid motion, any plane figure is carried rigidly from one position to another. If the rigid motion is a translation and if the origin O is carried into the point O', then any point P is carried into a point P' such that $\overline{PP'} = \overline{OO'}$. A translation is therefore completely determined by the vector

$\overline{OO'}$. In a translation, any straight line parallel to $\overline{OO'}$ is transformed into itself, although the individual points of such a straight line do not remain fixed; any other straight line is transformed into its parallel. If, on the other hand, the rigid motion is a rotation about the origin, the origin remains fixed and any straight line passing through the origin is rotated about the origin through a constant angle called the *angle of rotation*. *E.g.*, the angle of rotation θ of (3.5) is given by $\cos\theta = a$, $\sin\theta = b$.

Let us now enquire whether any point is left fixed by the rigid motion (3.1). If such a point (x, y) exists, we must have

$$x = ax + by + c_1, \qquad y = -bx + ay + c_2$$

Or

$$\begin{aligned} (a-1)x + by + c_1 &= 0 \\ -bx + (a-1)y + c_2 &= 0 \end{aligned}$$

Solution for (x, y) exists if the determinant of the coefficients

$$\begin{vmatrix} a-1 & b \\ -b & a-1 \end{vmatrix} = 2(1-a)$$

does not vanish. The solution is then given by

$$2(1-a)x = \begin{vmatrix} b & c_1 \\ a-1 & c_2 \end{vmatrix}, \qquad 2(1-a)\,y = \begin{vmatrix} a-1 & c_1 \\ -b & c_2 \end{vmatrix}$$

Such a point therefore remains fixed unless $a = 1$, that is, unless the rigid motion is a translation. When a point F remains fixed, the rigid motion is a rotation about F. Hence, *a rigid motion is either a translation or a rotation.*

11.1. Product of rigid motions. Let a rigid motion M_1 defined by

$$\begin{aligned} \bar{x} &= ax + by + c_1 \\ \bar{y} &= -bx + ay + c_2 \end{aligned} \qquad a^2 + b^2 = 1$$

be followed by another rigid motion M_2 defined by

$$\begin{aligned} x' &= a'\bar{x} + b'\bar{y} + d_1 \\ y' &= -b'\bar{x} + a'\bar{y} + d_2 \end{aligned} \qquad a'^2 + b'^2 = 1$$

Then the transformation leading from (x, y) to (x', y') is said to be the *product* or the *resultant* M_2M_1 and is given by

$$\begin{aligned} x' &= (aa' - bb')\,x + (ab' + ba')\,y + c_1a' + c_2b' + d_1 \\ y' &= -(ab' + ba')\,x + (aa' - bb')\,y - c_1b' + c_2a' + d_2 \end{aligned}$$

which is of the form (3.1). Since

$$\begin{vmatrix} aa' - bb' & ab' + ba' \\ -(ab' + ba') & aa' - bb' \end{vmatrix} = \begin{vmatrix} a & b \\ -b & a \end{vmatrix} \begin{vmatrix} a' & b' \\ -b' & a' \end{vmatrix} = 1,$$

the product M_2M_1 is a rigid motion. In general, *the product of any number of rigid motions is a rigid motion.* It should be noticed that, in general, $M_2M_1 \neq M_1M_2$, *i.e.*, the order of the factors cannot, in general, be altered without altering the product. A particular case of this fact is shown below. The choice of the notation M_2M_1 (instead of M_1M_2), as used above, is however a matter of convention.

Let R be the rotation (about the origin)

$$\bar{x}=ax+by, \qquad \bar{y}=-bx+ay,$$

and T be the translation

$$x'=\bar{x}+c_1, \qquad y'=\bar{y}+c_2$$

Then R followed by T, *i.e.*, the product TR, is the rigid motion M_1. But if T is followed by R, we must write the equations of T, R as

$$\begin{aligned} \bar{x}&=x+c_1 \\ \bar{y}&=y+c_2 \end{aligned} \quad \text{and} \quad \begin{aligned} x'&=\ \ a\bar{x}+b\bar{y} \\ y'&=-b\bar{x}+a\bar{y} \end{aligned}$$

respectively. Then the product RT does not result in the rigid motion M_1. This is expressed by saying that the product of rigid motions is not, in general, *commutative*. The product is, however, *associative*. It may be seen that the product of two translations or of two rotations about the same point is commutative.

CHAPTER IV

CONICS

12. Classifications of conics and their equations in normal forms. The general equation of the second degree may be written as

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0, \tag{4.1}$$

where the coefficients are constants of which a, b, c are not all zero. Curves satisfying this equation are curves of the second degree, usually known as *conics*. Consider the determinants

$$\Delta = \begin{vmatrix} a & b \\ b & c \end{vmatrix} \qquad \Phi = \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}$$

Let an arbitrary rigid motion be given by

$$\begin{aligned} x &= px' + qy' + r_1 \\ y &= -qx' + py' + r_2 \end{aligned} \qquad p^2 + q^2 = 1$$

Under this transformation, the equation (4.1) is transformed into

$$a'x'^2 + 2b'x'y' + c'y'^2 + 2d'x' + 2e'y' + f' = 0,$$

where

$$\begin{aligned} a' &= ap^2 - 2bpq + cq^2 \\ b' &= b(p^2 - q^2) + (a - c)pq \\ c' &= aq^2 + 2bpq + cp^2. \end{aligned}$$

Therefore, by calculation,

$$\Delta' = \begin{vmatrix} a' & b' \\ b' & c' \end{vmatrix} = \Delta, \qquad \Phi' = \begin{vmatrix} a' & b' & d' \\ b' & c' & e' \\ d' & e' & f' \end{vmatrix} = \Phi$$

Hence, Δ and Φ remain invariant under rigid motions. The quantity Φ is called the *discriminant* of the equation (4.1), and the vanishing of the discriminant is the necessary and sufficient condition that (4.1) should break up into two linear factors. The conic then consists of two straight lines and is said to be a *degenerate* conic.

We proceed to reduce the equation (4.1) to its *normal forms*.

I. Firstly, let $\Delta \neq 0$.

CALCUT
CENTRAL
LIBRARY
UNIVE

Apply a translation
$$x = x' + c_1$$
$$y = y' + c_2$$
so that the point (c_1, c_2) is the new origin. The equation (4.1) becomes
$$ax'^2 + 2bx'y' + cy'^2 + 2(ac_1 + bc_2 + d)\,x' + 2(bc_1 + cc_2 + e)\,y' + g = 0,$$
where
$$g = ac_1^2 + 2bc_1c_2 + cc_2^2 + 2dc_1 + 2ec_2 + f$$
Determine c_1, c_2 such that
$$ac_1 + bc_2 + d = 0$$
$$bc_1 + cc_2 + e = 0$$
Solution for c_1, c_2 exists, because $\Delta \neq 0$. The equation now reduces to
$$ax'^2 + 2bx'y' + cy'^2 + h = 0, \tag{4.1$'$}$$
where h is the expression obtained by substituting the values of c_1, c_2 in g. Now, if $P_1 = (x', y')$ is a point on the curve (4.1$'$), $P_2 = (-x', -y')$ is also a point on the same. But P_1 and P_2 are collinear with the new origin $O' = (c_1, c_2)$ and are equally distant from it. Accordingly, O' is called the *centre* of all conics satisfying (4.1$'$), and these conics are called *central conics*.

Next, apply a rotation
$$x' = px'' + qy'' \qquad\qquad p^2 + q^2 = 1$$
$$y' = -qx'' + py''$$
The equation (4.1$'$) takes the form
$$(ap^2 - 2bpq + cq^2)\,x''^2 + 2\{b(p^2 - q^2) + (a - c)pq\}\,x''y'' + (aq^2 + 2bpq + cp^2)\,y''^2 + h = 0$$
Let $p/q = \xi$ ($q \neq 0$, because the transformation is a rotation), and determine ξ so that the coefficient of $x''y''$ vanishes. So,
$$b\,(\xi^2 - 1) + (a - c)\,\xi = 0.$$
This is a quadratic equation in ξ whose roots ξ_1, ξ_2 satisfy
$$\xi_1\xi_2 = -1, \quad \text{or} \quad \cot\theta_1 \cot\theta_2 = -1,$$
where θ_1, θ_2 are the angles of rotation corresponding to the two values of ξ (see § 11). So,
$$\theta_1 \sim \theta_2 = m\pi/2,$$
where m is an odd integer. Therefore, the positive x''- and y''-axes corresponding to one value of ξ is obtained from the other by a rotation through $\pi/2$, and so the two sets of axes corresponding to the two values of ξ consist of the same two straight lines. Thus, the equation (4.1$'$) ultimately reduces (writing x, y for x'', y'') to the form
$$Ax^2 + By^2 + C = 0, \quad AB \neq 0 \tag{4.1$''$}$$

We now proceed to consider the different cases that may arise :

(1) $C=0$.

(*i*) $AB>0$. In this case A and B may, without loss of generality, be considered positive and therefore we may put

$$A=1/a^2, \qquad B=1/b^2,$$

where a, b are two real quantities (not to be confused with the coefficients a, b in (4.1)). The equation (4.1″) takes the normal form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0 \tag{4.2}$$

The curve passes through only one real point (0, 0). The curve is called a *null ellipse*.

(*ii*) $AB<0$. In this case we may put

$$A=1/a^2, \quad B=-1/b^2$$

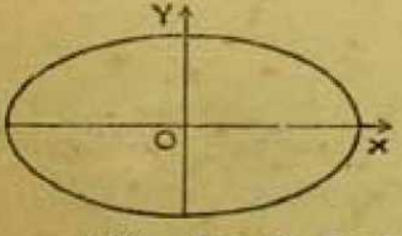
and obtain the normal form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \tag{4.3}$$

The curve consists of *two intersecting straight lines* $x/a \pm y/b=0$.

(2) $C\neq 0$. Without loss of generality we may suppose $C=-1$.

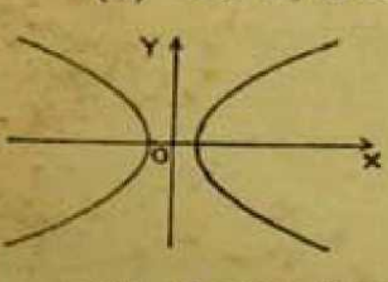
(*i*) $A>0$, $B>0$. The equation now takes the normal form



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{4.4}$$

The curve is called an *ellipse* (here $\Phi<0$).

(*ii*) $A>0$, $B<0$. We have here the normal form



$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \tag{4.5}$$

The curve is called a *hyperbola*.

(*iii*) $A<0$, $B<0$. The normal form is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1 \tag{4.6}$$

There is no real point satisfying the equation. The curve is a *nondegenerate conic without real trace* (here $\Phi>0$).

II. Secondly, let $\Delta=0$, or $b=\pm\sqrt{ac}$

As a and c have the same sign we may, without loss of generality, suppose that they are positive. Hence, we may apply the rotation

$$x' = \frac{\sqrt{a}}{\sqrt{a+c}}x \pm \frac{\sqrt{c}}{\sqrt{a+c}}y$$

$$y' = \mp \frac{\sqrt{c}}{\sqrt{a+c}}x + \frac{\sqrt{a}}{\sqrt{a+c}}y$$

Choose the upper or the lower sign according as b is positive or negative. Then

$$(a+c)\,x'^2 = ax^2+2bxy+cy^2$$

Therefore, the equation (4.1) is transformed into

$$x'^2+l(x', y')=0,$$

where $l(x', y')$ is a linear function, equal to $2kx'+2my'+r$, say. Again, apply the translation

$$x''=x'+k$$

$$y''=y'$$

in order to get rid of the term involving the first power of x'. The equation (4.1) now takes the form

$$x''^2+2my''+n=0 \qquad (4.1''')$$

The different cases that may arise are :

(1) $m = 0$.

(*i*) $n \neq 0$. The equation (4.1''') reduces to $x''^2+n=0$. Put $n=\pm a^2$ (not to be confused with the a above) according as n is positive or negative. So, we obtain the normal forms (writing x for x'')

$$x^2+a^2=0, \qquad (4.7)$$

a pair of parallel straight lines without real trace, and

$$x^2-a^2=0, \qquad (4.8)$$

a pair of parallel straight lines.

(*ii*) $n=0$. Here we have the normal form

$$x^2=0, \qquad (4.9)$$

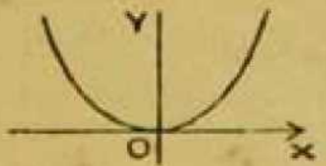
representing *two coincident straight lines.*

(2) $m \neq 0$. In this case, if $n \neq 0$, we make the further transformation

$$\bar{x}=x''$$

$$\bar{y}=y''+n/2m$$

The equation (4.1‴) now takes the normal form (writing x, y for $\bar{x}$, $\bar{y}$)



$$x^2+2my=0 \tag{4.10}$$

The curve is called a *parabola*.

We have dealt with all the different cases that may arise, and there is no other type of curve of the second degree. We divide all conics into three classes : (1) *hyperbolic*, for which $\Delta<0$, (2) *parabolic*, for which $\Delta=0$ and (3) *elliptic*, for which $\Delta>0$. The *normal forms* of the equations of the three classes of conics are shown in the following table :

Hyperbolic, $\Delta<0$	Parabolic, $\Delta=0$	Elliptic, $\Delta>0$
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$x^2+2my=0$ $x^2-a^2=0$ $x^2+a^2=0$ $x^2=0$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$

We have thus arrived at the following conclusion :

By choice of the coordinate system, equation (4.1) *of an arbitrary conic can be transformed into one and only one of the above normal forms.*

Hereafter, by a conic we shall mean a conic with a real trace only unless otherwise stated.

Application 1. Reduce the following equation to its normal form :

$$5x^2-2xy+5y^2-8x-8y-8=0.$$

Here $\Delta>0$. So, apply first an arbitrary translation and choose the new origin so that the linear terms x', y' drop out in the transformed equation. The coordinates of the new origin are seen to be (1, 1) and the transformed equation is

$$5x'^2-2x'y'+5y'^2-16=0.$$

Then apply an arbitrary rotation about the origin and choose an angle of rotation so that the coefficient of $x''y''$ in the transformed equation vanishes. The angle may be chosen as $\pi/4$, and transformed equation is

$$6x''^2 + 4y''^2 - 16 = 0$$

Hence the normal form is

$$\frac{x^2}{8/3} + \frac{y^2}{4} = 1$$

The curve is an ellipse.

Application 2. Determine the different kinds of conics represented by the equation

$$x^2 + 4\lambda xy + 4y^2 + 2(1+\lambda)x + 8y + 5 + 2\lambda = 0$$

as λ changes from large positive value to large negative value. Examine, in particular, the critical cases $\lambda = 1, 0, -1, -2$. (Pembroke, 1911)

Here

$$\Delta = 4\,(1-\lambda^2)$$

$$\Phi = -8\lambda\,(\lambda^2 + \lambda - 2)$$

So

$\Phi = 0$ when $\lambda = +1, 0, -2$

Case $\Delta < 0$: $\lambda > +1, \lambda < -1$

$\lambda < -1$ may be divided into three intervals

$-2 < \lambda < -1, \lambda = -2, \lambda < -2$

$\lambda = -2$ makes $\Phi = 0$ and therefore gives a degenerate conic.

Case $\Delta = 0$: $\lambda = +1, -1$

$\lambda = +1$ makes $\Phi = 0$, puts the given equation as $(x+2y+2)^2 + 3 = 0$ and therefore gives a pair of parallel straight lines without real trace.

Case $\Delta > 0$: $0 < \lambda < 1, \lambda = 0, -1 < \lambda < 0$

$\lambda = 0$ makes $\Phi = 0$ and therefore gives null ellipse.

$-1 < \lambda < 0$ makes $\Phi < 0$ and therefore gives ellipse.

We have therefore the following result :

(1) $\lambda > 1$, hyperbola ;
(2) $\lambda = 1$, a pair of parallel straight lines without real trace ;
(3) $0 < \lambda < 1$, nondegenerate conic without real trace ;
(4) $\lambda = 0$, null ellipse ;
(5) $-1 < \lambda < 0$, ellipse ;
(6) $\lambda = -1$, parabola ;
(7) $-2 < \lambda < -1$, hyperbola ;
(8) $\lambda = -2$, a pair of intersecting straight lines ;
(9) $\lambda < -2$, hyperbola.

13. Pole and polar. Tangent. Consider a nondegenerate conic given by the general equation

$$F(x, y) \equiv ax^2+2bxy+cy^2+2dx+2ey+f=0$$

Let $P=(x, y)$ be a point given by

$$\begin{aligned} x&=x_1+\rho p \\ y&=y_1+\rho q \end{aligned} \qquad p^2+q^2=1$$

on a straight line through a given point $P_1=(x_1, y_1)$. So take $\overline{P_1P}=\rho$. The points of intersection of the straight line and the conic are given by

$$A\rho^2+2B\rho+C=0, \qquad \text{.. (4.11)}$$

where
$$\begin{aligned} A&=ap^2+2bpq+cq^2 \\ B&=(ax_1+by_1+d)p+(bx_1+cy_1+e)q \\ C&=F(x_1, y_1) \end{aligned}$$

We suppose that P_1 is neither a point on the conic (so that $C\neq 0$) nor the centre of the conic, in case the conic is a central conic (so that $B\neq 0$). We consider only those straight lines through P_1 that intersect the conic in two points P', P'' corresponding to the two roots ρ_1, ρ_2 of the equation (4.11) (so that $A\neq 0$).

We have
$$\rho_1+\rho_2=-2B/A, \quad \rho_1\rho_2=C/A$$

Also
$$\overline{P_1P'}=\rho_1, \qquad \overline{P_1P''}=\rho_2$$

Let the point P be the harmonic conjugate of P_1 with respect to P', P'', *i.e.*, let

$$(P_1P, P'P'')=-1.$$

So
$$\frac{\overline{P_1P'}.\overline{PP''}}{\overline{P_1P''}.\overline{PP'}}=-1, \qquad \text{or} \quad \frac{\rho_1(\rho_2-\rho)}{\rho_2(\rho_1-\rho)}=-1$$

Accordingly
$$\rho=\frac{2\rho_1\rho_2}{\rho_1+\rho_2}=-\frac{C}{B}$$

Therefore, the coordinates of P are given by

$$\begin{aligned} x&=x_1-pC/B \\ y&=y_1-qC/B \end{aligned} \qquad p^2+q^2=1.$$

Eliminate p, q between these equations (multiply the two equations by ax_1+by_1+d and bx_1+cy_1+e respectively and add) and obtain

$$(x-x_1)\,(ax_1+by_1+d)+(y-y_1)\,(bx_1+cy_1+e)+F(x_1, y_1)=0$$

Or
$$(ax_1+by_1+d)\,x+(bx_1+cy_1+e)\,y+dx_1+ey_1+f=0 \qquad (4.12)$$

This is a linear equation in which the coefficients of x and y cannot both vanish, because $B \neq 0$, and so the equation represents a straight line. Therefore given P_1, the point P always lies on a fixed straight line. This straight line, (4.12), is called the *polar* of P_1 with respect to the given conic ; the point P_1 is called the *pole*.

We may state the above result in the following manner :

Let a conic and a point P_1 be given, where P_1 is neither on the conic nor its centre (in the case of a central conic). If a straight line be drawn through P_1 to meet the conic in P', P'', then the locus of a point P on the straight line such that $(P_1P, P'P'')=-1$ is a straight line called the polar of P_1 with respect to the conic : the point P_1 is called the pole of the polar.

It may be mentioned that not every point P_2 of this locus has necessarily the above property ; the straight line P_1P_2 may not at all intersect the conic. The equation (4.12) represents a straight line except in the case where

$$ax_1+by_1+d=0, \qquad bx_1+cy_1+e=0$$

that is, except in the case of P_1 being the centre.

We shall now consider the geometrical interpretation of the line (4.12) when P_1 is a point on the conic. Applying the method of differential calculus, we may express (4.12) as

$$(x-x_1)\left(\frac{\partial F}{\partial x}\right)_{x_1, y_1}+(y-y_1)\left(\frac{\partial F}{\partial y}\right)_{x_1, y_1}+F(x_1, y_1)=0$$

If $C=F(x_1, y_1)=0$, this line is called the *tangent* to the conic at P_1. For this tangent,

$$p/q=-\left(\frac{\partial F}{\partial y}\right)_{x_1, y_1} \Big/ \left(\frac{\partial F}{\partial x}\right)_{x_1, y_1}$$

and therefore $B=0$ also. Hence the equation (4.11) has no root other than $\rho=0$. The tangent is therefore geometrically distinguished by the property that it intersects the conic in one point only ; but it is not completely characterised by this property. For, the equation (4.11) has $\rho=0$ as the only root if either $B=C=0$, $A\neq 0$, or $A=C=0$, $B\neq 0$. Hence, the tangent at P_1 is the limiting position of a line intersecting the conic in P_1 and P_2 when P_2 approaches and ultimately coincides with P_1, and is therefore geometrically different from other lines. So we can expect that, after a change in the system of coordinates, the tangent will appear as a line intersecting the conic in two ultimately coincident points.

The reader may verify this result as an exercise by applying the formula of rigid motion and showing that a tangent to a conic is transformed by an arbitrary rigid motion into a tangent to the transformed conic.

We now adopt by *definition* that *the polar of a point on a conic is the tangent to the conic at the point.* Therefore (4.12) is the equation of the polar of P_1, if it at all represents a straight line.

Let $P_2=(x_2, y_2)$ be an arbitrary point on the polar of P_1; then

$$ax_1x_2+b(x_1y_2+x_2y_1)+cy_1y_2+d(x_1+x_2)+e(y_1+y_2)+f=0. \tag{4.13}$$

As this condition is symmetrical in the coordinates of P_1 and P_2, it follows that P_1 is situated on the polar P_2. The points P_1 and P_2 are then said to be *conjugate points* with respect to the given conic, *e.g.*, the points conjugate to a point P of the conic are the points of the tangent at P. Similarly, two lines are *conjugate lines* with respect to the conic when one of them passes through the pole of the other.

If the polars p_3 and p_4 of two points P_3 and P_4 intersect in P_1, then P_1 is conjugate to P_3 and P_4; hence the polar p_1 of P_1 passes through P_3 and P_4 and is therefore the straight line joining P_3 and P_4. If p_3 and p_4 are parallel, there exists no pole of the straight line P_3P_4 which then passes through the centre of the conic; hence the polars of all points of P_3P_4 (except the centre of the conic) are parallel. If p_1 intersects the conic in two points P' and P'', the polars p' and p'' of P' and P'' must intersect in the pole P_1 of p_1; if, conversely, the tangents at two points P' and P'' intersect in $P_1=(x_1, y_1)$, this point is the pole of the straight line $P'P''$. Now

$$Q(x, y)\equiv\{(ax_1+by_1+d)(x-x_1)+(bx_1+cy_1+e)(y-y_1)\}^2$$
$$-\{a(x-x_1)^2+2b(x-x_1)(y-y_1)+c(y-y_1)^2\}F(x_1, y_1)=0 \tag{4.14}$$

is the equation of a curve of second degree with centre P_1, and passing through it. Hence, it is degenerate and is either a null ellipse or a pair of straight lines intersecting in P_1. But

$$Q(x, y)=\{(ax_1+by_1+d)x+(bx_1+cy_1+e)y+dx_1+ey_1+f\}^2$$
$$-F(x, y)F(x_1, y_1)$$

This shows that the degenerate curve passes through the points of intersection of the given curve and the polar of P_1. The equation (4.14) represents therefore the pair of tangents drawn from P_1 to the conic, if these tangents exist.

Finally, let us consider the case of straight lines which are not tangents but intersect the conic in one point only. It has been shown that in this case $A=C=0$, $B\neq0$. As this property has been proved to be independent of the manner in which we choose the system of coordinates, there is no loss of generality to consider normal forms only.

(1) In the case of a hyperbola $x^2/a^2-y^2/b^2=1$, the condition $A=0$ becomes

$$\left(\frac{p}{a}\right)^2-\left(\frac{q}{b}\right)^2=0, \quad \text{or,} \quad \frac{p}{q}=\pm\frac{a}{b}$$

Hence, there are two straight lines through P_1 which will not meet the curve again. The two straight lines through the centre (the origin) parallel to these two lines are called the *asymptotes* of the hyperbola. So, the asymptotes do not meet the hyperbola in any point.

(2) In the case of a parabola $x^2+2my=0$, the condition $A=0$ becomes

$$p=0$$

Hence, there is one straight line through P_1 which will not meet the curve again. The straight line through the vertex (the origin) parallel to this line is called the *axis* of the parabola.

(3) In the case of an ellipse $x^2/a^2+y^2/b^2=1$, the condition $A=0$ becomes

$$\left(\frac{p}{a}\right)^2+\left(\frac{q}{b}\right)^2=0$$

Since p, q cannot both be zero, there is no (real) straight line through P_1 that will not meet the curve again. An ellipse is therefore a closed curve.

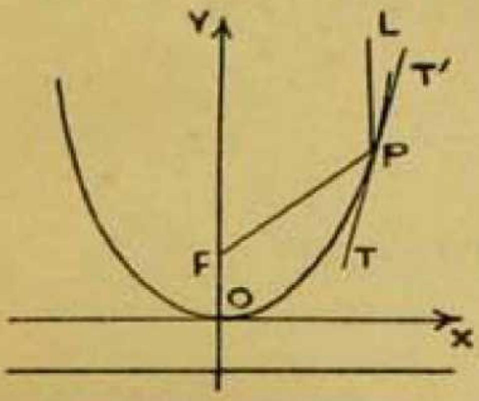
We may sum up the above results in the following way :

If through any point on a nondegenerate conic we draw such straight lines as will not meet the curve in a second distinct point, then we obtain the tangent and straight lines parallel to the asymptotes in the case of a hyperbola, the tangent and the straight line parallel to the axis in the case of a parabola and only the tangent in the case of an ellipse.

14. Focus and directrix. A nondegenerate conic is also defined as the locus of a point whose distance from a fixed point, called the *focus*, is in a constant ratio to its perpendicular distance from a fixed straight line, called the *directrix*. The conic is a parabola, an ellipse or a hyperbola according as the constant ratio, called the *eccentricity*, is equal to, less than or greater than unity. Let the eccentricity be denoted by the positive constant e.

Parabola, $e=1$. If the axes are so chosen that the focus is the point $(0, -m/2)$ and the directrix is the straight line $y-m/2=0$, we obtain the normal form (4.10). In this case, the y-axis is called the *axis* of the parabola.

Let P be a point of a parabola of which F is the focus ; also let TPT' be the tangent at P and PL the straight line parallel to the axis. It follows from the definition of a parabola given above that

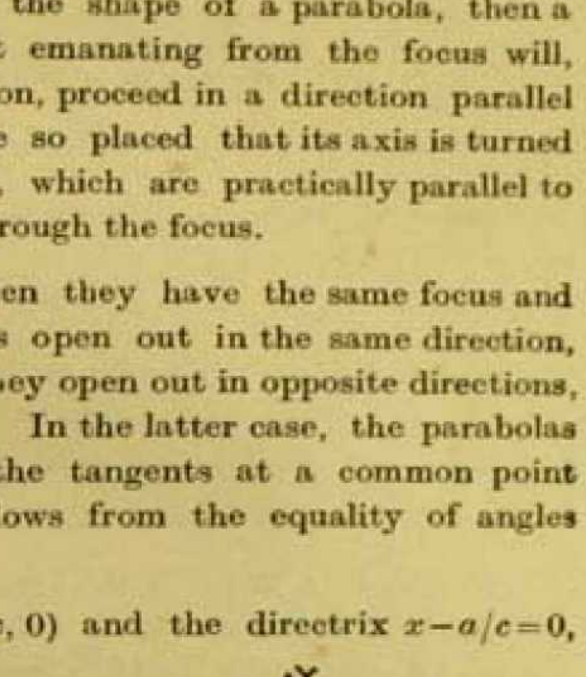


$$\sphericalangle TPF = \sphericalangle LPT',$$

as in the figure. This property gives rise to what may be called the *optical property of a parabola* : If we suppose that we have a reflector of the shape of a parabola, then a ray of light emanating from the focus will, after reflection, proceed in a direction parallel to the axis. Or, if such a reflector be so placed that its axis is turned towards the sun, then the sun's rays, which are practically parallel to one another, will, after reflection, pass through the focus.

Parabolas are said to be *confocal* when they have the same focus and the same axis. If two confocal parabolas open out in the same direction, they have no point in common ; but if they open out in opposite directions, they intersect one another in two points. In the latter case, the parabolas cut one another orthogonally ; that is, the tangents at a common point are orthogonal to one another. This follows from the equality of angles given above.

Ellipse, $c < 1$. If the focus is $(ac, 0)$ and the directrix $x-a/c=0$, we obtain the normal form (4.4), where $b^2=a^2(1-c^2)$. Without loss of generality we may suppose a, b positive. The ellipse has a second focus $(-ac, 0)$ and a second corresponding directrix $x+a/c=0$. In the figure, the segments AA', BB' are *the major* and *the minor axes* whose lengths are $2a, 2b$ respectively. The axes intersect one another in the centre (the origin) and meet the curve in the *vertices*.



If we eliminate θ between the equations

$$\begin{aligned} x &= a \cos \theta \\ y &= b \sin \theta \end{aligned} \tag{4.15}$$

we obtain the same equation (4.4) of the ellipse. So, the above equations may be taken as the *parametric equations* of the ellipse, θ being the parameter.

Let $P=(x, y)$ be a point of the ellipse (4.15) of which the two foci are F, F', and N the foot of the perpendicular from P on the major axis. So, as in the figure,

$$\overline{ON}=a\cos\theta,\quad \overline{NP}=b\sin\theta,\quad \overline{OF}=ac$$

Then

$$\begin{aligned} & |\, FP \,| + |\, F'P \,| \\ &= |\, \sqrt{\{(a\cos\theta-ac)^2+b^2\sin^2\theta\}} \,| + |\, \sqrt{\{(a\cos\theta+ac)^2+b^2\sin^2\theta\}} \,| \\ &= |\, a-ac\cos\theta \,| + |\, a+ac\cos\theta \,| \\ &= 2\,|\, a \,|\,,\ \text{the length of the major axis.} \end{aligned}$$

This property may also be taken as a definition of an ellipse.

Let TPT' be the tangent to the ellipse at P. Then it may be seen that

$$\sphericalangle\, TPF = \sphericalangle\, F'PT'.$$

This property leads to what may be called *the optical property of an ellipse* : If we have a reflector of the shape of an ellipse, then a ray of light proceeding from one of the foci will, after reflection, pass through the other focus.

Hyperbola, $c > 1$. If the focus is $(ac, 0)$ and the directrix $x-a/c=0$, we obtain the normal form (4.5), where $b^2=a^2(c^2-1)$. The hyperbola has a second focus $(-ac, 0)$ and a second corresponding directrix $x+a/c=0$. In the normal form, the axes of coordinates are the *axes* of the hyperbola intersecting in the centre. Let us take

$$x/a=\mu+\nu,\quad y/b=\mu-\nu$$

In order that these equations shall represent the hyperbola, we must have $4\mu\nu=1$. So, put $\mu=t/2$, $\nu=1/2t$. Therefore the *parametric equations* of the hyperbola are

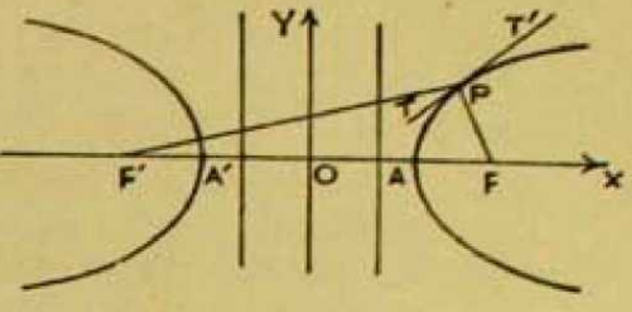
$$\begin{aligned} x&=a\,(t^2+1)/2t \\ y&=b\,(t^2-1)/2t, \end{aligned}$$

t being a parameter, not equal to zero. We obtain the two branches of the curve for $t \gtrless 0$. These equations can also be written as

$$\left.\begin{aligned} x&=a\sec\theta \\ y&=b\tan\theta \end{aligned}\right. \qquad \text{(4.16}$$

where θ is now the parameter.

Let P be a point on the curve, of which the two *foci* are F, F' and the two *vertices* are A, A'. Then, as in the case of ellipse, the difference between $|FP|$ and $|F'P|$ = the difference between $|\sqrt{\{(a \sec\theta - ac)^2 + b^2\tan^2\theta\}}|$ and $|\sqrt{\{(a \sec\theta + ac)^2 + b^2\tan^2\theta\}}|$ = the difference between $|a - ac\sec\theta|$ and $|a + ac\sec\theta|$. The absolute value of this difference



$$= 2\,|a| = |AA'|$$

This property may also be taken as the definition of a hyperbola.

Consider the straight line $x/a + y/b = 0$. In Hessian normal form, the equation is

$$l(x, y) \equiv \frac{x}{a\sqrt{(1/a^2 + 1/b^2)}} + \frac{y}{b\sqrt{(1/a^2 + 1/b^2)}} = 0$$

Therefore, the perpendicular distance of a point (x, y) from the straight line is $l(x, y)$. If (x, y) is a point on the hyperbola,

$$x^2/a^2 - y^2/b^2 = 1, \quad \text{or} \quad x/a + y/b = \frac{1}{x/a - y/b},$$

or

$$l(x, y) = \frac{1}{(x/a - y/b)\sqrt{(1/a^2 + 1/b^2)}}$$

So, the perpendicular distance tends to zero as x tends to $\pm\infty$ and y tends to $\mp\infty$. Similarly, the perpendicular distance of a point on the hyperbola from the straight line $x/a - y/b = 0$ tends to zero as x, y tends to $\pm\infty$. These two lines $x/a \mp y/b = 0$ are the asymptotes of the hyperbola. If the asymptotes are orthogonal to one another, we have $a = b$ and the hyperbola is then called a *rectangular* (*or equilateral*) *hyperbola*. The hyperbola $y^2/b^2 - x^2/a^2 = 1$, whose foci are on the y-axis, is said to be *conjugate* to the given hyperbola.

If TPT' be the tangent to the hyperbola at a point P, it may be seen that $\measuredangle FPT = \measuredangle F'PT$, as in the figure.

Ellipses and hyperbolas are said to be *confocal* when they have the same foci. If an ellipse and a hyperbola are confocal, they intersect orthogonally. This follows from the equality of angles given above.

Lastly, it is seen from the equations of directrices of nondegenerate conics that *a directrix is the polar of the corresponding focus with respect to the conic.*

14.1. The complex plane. It has been shown that we can represent the points of the plane by pairs of real numbers in such a manner that the points of a straight line are represented by the solutions of a linear equation, the points of a conic by the solutions of a quadratic equation, etc. On classifying the quadratic equations we have seen that some of them, *e.g.*,

$$x^2+y^2+1=0,$$

do not correspond to conics as there is no pair of real numbers (x, y) satisfying the equation. The equations

$$ax^2+by^2=0 \quad a>0, \quad b>0$$

are only satisfied by $(x, y)=(0, 0)$ independent of the choice of the positive values of a and b. Although these "conics" are equal in the sense that they consist of the same set of points (namely, the origin only), they are expressed by different equations and there exist, in general, no rigid motions transforming them into one another. It will be shown later on that these equations have different interpretations, as each of them is connected with a certain bilinear form generating a "polar-field".

In this article we shall consider the matter from another point of view by applying complex numbers and introducing complex points. Let us *define* a *complex point* as an ordered pair (x, y) of complex numbers. The complex points satisfying a complex linear equation

$$ax+by+c=0, \quad (a, b) \neq (0, 0) \tag{4.17}$$

shall then represent a *complex straight line*, and the complex points satisfying an equation of the second degree a *complex conic*. If, in particular, both values x and y are real, the complex point (x, y) will be a *real point*. If moreover a, b, c are real, the straight line (4.17) will be a *real straight line*. Every real straight line contains complex points which are not real; for if (x, y) is a point of (4.17), $(x+bi, y-ai)$ is also a point of (4.17). The set of all complex points is the *complex plane*, and the subset of all real points in it is the *real plane*.

The geometry of complex plane is in some respect of a simpler structure than the ordinary (real) plane geometry; *e.g.*, conics and straight lines always intersect in the complex plane. But it has also some special features which seem to be rather paradoxical at the first sight. We have however to state at the outset that *theorems which are valid in the ordinary plane geometry cannot be applied to the complex plane without a proof.* Although we do not intend to go into details, we need, besides points, straight lines and conics, the notions of distance and angle in the complex plane. We *define* them in the following manner:

I. The square of the "distance" between two points $P_1=(x_1, y_1)$ and $P_2=(x_2, y_2)$ is equal to

$$|P_1P_2|^2=(x_1-x_2)^2+(y_1-y_2)^2 \tag{4.18}$$

This expression therefore will not change when P_1 and P_2 are interchanged. For real points, this definition agrees with the usual notion of distance. In any case, the square of the distance is uniquely defined by (4.18), but it may also be a negative or a non-real number. If the distance is equal to zero, the points P_1, P_2 are not necessarily coincident.

II. The "angle" θ between two straight lines

$$y=mx+\gamma, \quad y=nx+\lambda$$

is defined by

$$\tan\theta=\frac{1}{i}\,\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}=\frac{n-m}{1+nm} \tag{4.19}$$

This θ is not defined for every pair of lines (*e.g.*, we shall notice shortly a case where it does not exist), and it is in no case uniquely defined, as an arbitrary multiple of π may be added. For real straight lines, (4.19) agrees with the usual notion.

Now, the locus of a point which is at a constant distance r from a fixed point (x_0, y_0) is given by

$$(x-x_0)^2+(y-y_0)^2=r^2$$

If $r=0$, the locus consists of two imaginary straight lines

$$\begin{aligned}(x-x_0)+i(y-y_0)&=0\\(x-x_0)-i(y-y_0)&=0\end{aligned} \qquad i^2=-1$$

These straight lines and their parallels are called *isotropic lines*. The isotropic lines are evidently lines without real trace in a real plane and there are two such lines through every point. There are therefore two kinds of isotropic lines and they form two systems of parallel straight lines. They are

$$\begin{matrix} x+iy+\mu=0 & & y-ix+\gamma=0 \\ & \text{or} & \\ x-iy+v=0 & & y+ix+\lambda=0 \end{matrix} \tag{4.20}$$

where μ, v, γ, λ are arbitrary constants. The isotropic lines have the following peculiarities :

(1) The distance between any two points on an isotropic line is zero. For, if (ξ_1, η_1), (ξ_2, η_2) are two points on the isotropic line $y-ix+\gamma=0$,

we have $\qquad \eta_1-i\xi_1+\gamma=0, \quad \eta_2-i\xi_2+\gamma=0,$

or $\qquad (\eta_1-\eta_2)-i(\xi_1-\xi_2)=0$

Therefore $\qquad (\eta_1-\eta_2)^2+(\xi_1-\xi_2)^2=0,$

which shows that the distance between the two points is zero.

(2) The angle between two isotropic lines and that between an isotropic line and any other line is trigonometrically undefined.

(*i*) In the case of two isotropic lines of the same kind,

$$y-ix+\gamma_1=0,\quad y-ix+\gamma_2=0,$$

$$\tan\theta=0/0 \quad\text{and}\quad \text{therefore } \theta \text{ is undefined.}$$

(*ii*) In the case of two isotropic lines of different kinds,

$$y+ix+\gamma=0,\quad y-ix+\lambda=0,$$
$$\tan\theta=2i/2=i$$

But $$\tan\theta=\frac{1}{i}\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}=\frac{1}{i}\frac{e^{2i\theta}-1}{e^{2i\theta}+1}$$

So $$i=\frac{1}{i}\frac{e^{2i\theta}-1}{e^{2i\theta}+1}$$

Therefore $e^{i\theta}=0$. Hence, θ is undefined.

(*iii*) In the case of an isotropic line and any other line,

$$y+ix+\gamma=0,\quad y-mx+\lambda=0,$$

$$\tan\theta=\frac{m+i}{1-im}=\frac{i(m+i)}{m+i}=i$$

We have here the same result as in (*ii*).

Now, the parabola $x^2+2my=0$ has the focus $(0,\,-m/2)$ and the directrix $y-m/2=0$. Consider the two isotropic lines

$$x=i(y+m/2),\qquad x=-i(y+m/2).$$

These two lines intersect one another at the point given by

$$x=0,\ y=-m/2\ ;$$

that is, they intersect at the focus. Moreover, these lines are tangents to the parabola and they touch the curve at the points given by

$$y=m/2,\ x=im \quad\text{and}\quad y=m/2,\ x=-im$$

respectively ; that is, they touch at the points where the directrix meets the parabola. Thus, *there are two isotropic tangents to a parabola ; they intersect at the focus and their points of contact lie on the directrix.*

The ellipse $x^2/a^2+y^2/b^2=1$ has two *real* foci $F,\ F'=(\pm ac,\,0)$ and two corresponding *real* directrices $x\mp a/c=0$, where the eccentricity c is given by $c=\surd(1-b^2/a^2)$. It may be seen that the ellipse has also two *imaginary* foci $G,\ G'=(0,\,\pm\, iac)$ and the two corresponding *imaginary* directrices $y=\pm a(1-c^2)/ic$, where the eccentricity is $ic/\surd(1-c^2)$. For, the equation of the ellipse can be written as

$$x^2+(y\mp iac)^2=-\{c^2/(1-c^2)\}\{y\mp a(1-c^2)/ic\}^2,$$

which reduces to the given form $x^2/a^2+y^2/b^2=1$.

As in the case of a parabola, it may be seen that the two isotropic lines $y=i\,(x-ac)$, $y=-i\,(x-ac)$ intersect one another at the focus $(ac, 0)$ and touch the ellipse at the points where the corresponding directrix $x-a/c=0$ meets the curve. Similarly the two isotropic lines $y=i(x+ac)$, $y=-i(x-ac)$ are tangents to the ellipse ; they intersect in the imaginary focus $(0, iac)$ and their points of contact lie on the corresponding imaginary directrix $y=a(1-c^2)/ic$. Thus, *there are four isotropic tangents to an ellipse which form two pairs of parallel lines ; any two non-parallel isotropic tangents intersect in a focus and their points of contact lie on the corresponding directrix.* The correspondence of focus, directrix and isotropic tangents of the ellipse $x^2/a^2+y^2+b^2=1$ is shown in the following table.

Eccentricity	Focus	Directrix	Pairs of isotropic tangents through the focus.
$c=\sqrt{(1-b^2/a^2)}$	$ac, 0$	$x=a/c$	$y=i(x-ac)$ $y=-i(x-ac)$
	$-ac, 0$	$x=-a/c$	$y=i(x+ac)$ $y=-i(x+ac)$
$ic/\sqrt{(1-c^2)}$	$0, iac$	$y=a(1-c^2)/ic$	$y=i(x+ac)$ $y=-i(x-ac)$
	$0, -iac$	$y=-a(1-c^2)/ic$	$y=i(x-ac)$ $y=-i(x+ac)$

It may be verified that the sum of the squares of the reciprocals of the two eccentricities is equal to unity.

Finally, let P be a point on the ellipse. Then from the parametric equations (4.15) of the ellipse it follows that the square of the length GP

$$=a^2\cos^2\theta+(b\sin\theta-iac)^2=(b-iac\sin\theta)^2\ ;$$

the square of the length $G'P$ is similarly $(b+iac\sin\theta)^2$. Therefore the sum of the lengths

$$GP+G'P=2b.$$

The case for a hyperbola is similar to that for an ellipse given above.

Laguerre's definition of angle. Let us take four straight lines p_1, p_2, p_I, p_J passing through a point, say the origin, the first two being ordinary straight lines and the last two isotropic lines. Let the equations of the four lines be $y=m_1x$, $y=m_2x$, $y=-ix$, $y=ix$ respectively. Let an ordinary straight line $x=h$ meet the four lines in the points P_1, P_2, P_I, P_J respectively. So

$$\overline{P_IP_1}=(m_1+i)h,\quad \overline{P_IP_2}=(m_2+i)h,\quad \overline{P_JP_1}=(m_1-i)h,\quad \overline{P_JP_2}=(m_2-i)h.$$

Therefore the cross-ratio

$$(p_1p_2,\ p_Ip_J)=(P_1P_2,\ P_IP_J)=\frac{(m_1+i)\ (m_2-i)}{(m_2+i)\ (m_1-i)}$$

$$=\frac{(1+m_1m_2)+i(m_2-m_1)}{(1+m_1m_2)-i(m_2-m_1)}=\frac{1+i(m_2-m_1)/(1+m_1m_2)}{1-i(m_2-m_1)/(1+m_1m_2)}$$

Let θ be the angle from p_1 to p_2 as defined by (4.17). Then

$$\tan\theta=(m_2-m_1)/(1+m_1m_2)$$

So $$(p_1p_2,\ p_Ip_J)=(1+i\tan\theta)/(1-i\tan\theta)=\frac{e^{i\theta}/\cos\theta}{e^{-i\theta}/\cos\theta}=e^{2i\theta}$$

Therefore $$\theta=\frac{1}{2i}\log\ (p_1p_2,\ p_Ip_J) \tag{4.21}$$

This is the definition of an angle θ from a straight line p_1 to another straight line p_2. If the two straight lines p_1, p_2 are orthogonal to one another, $\sin 2\theta=0$, $\cos 2\theta=-1$, and therefore

$$(p_1p_2,\ p_Ip_J)=-1$$

Thus, *two straight lines are orthogonal to one another when they are harmonically separated by the two isotropic lines passing through their common point.*

If θ vanishes, $(p_1p_2,\ p_Ip_J)=1$ and therefore $m_1=m_2$, as is to be expected.

CHAPTER V

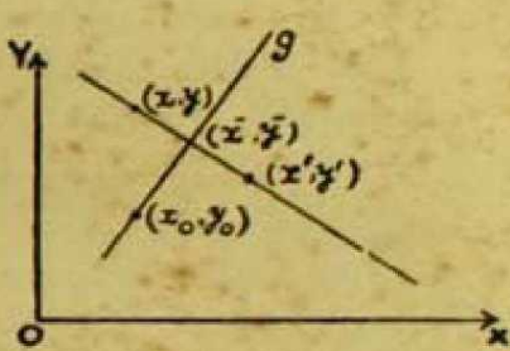
TRANSFORMATIONS OF SYMMETRY AND SIMILARITY

15. Symmetry. Let a straight line g be defined by

$$\begin{aligned} x &= x_0 + \rho p \\ y &= y_0 + \rho q \end{aligned} \qquad p^2 + q^2 = 1$$

Then an arbitrary point $P = (x, y)$ of the plane is given by

$$x = x_0 + \rho p - \sigma q, \qquad y = y_0 + \rho q + \sigma p$$



The point $P' = (x', y')$ given by

$$x' = x_0 + \rho p + \sigma q, \qquad y' = y_0 + \rho q - \sigma p$$

is such that the segment joining the points P, P' is bisected orthogonally by g.

Since $x' - x = 2\sigma q$, $y' - y = -2\sigma p$

and $(x - x_0)q - (y - y_0)p = -\sigma$,

so

$$\begin{aligned} (x' - x) &= -2(x - x_0)q^2 + 2(y - y_0)pq \\ (y' - y) &= 2(x - x_0)pq - 2(y - y_0)p^2 \end{aligned}$$

Or

$$\begin{aligned} x' &= (p^2 - q^2)x + 2pqy + 2q(qx_0 - py_0) \\ y' &= 2pqx - (p^2 - q^2)y - 2p(qx_0 - py_0) \end{aligned}$$

This is a linear transformation of the coordinates (x, y) to (x', y'). In order to write it in a suitable form, put

$$p = \cos\theta, \quad q = \sin\theta, \quad qx_0 - py_0 = \omega$$

So, the transformation can be written as

$$\begin{aligned} x' &= x\cos 2\theta + y\sin 2\theta + 2\omega\sin\theta \\ y' &= x\sin 2\theta - y\cos 2\theta - 2\omega\cos\theta \end{aligned} \tag{5.1}$$

and the equation of g as

$$x\sin\theta - y\cos\theta = \omega \tag{5.1'}$$

The transformation (5.1) is called an *orthogonal reflexion in the line* (5.1').

In particular, orthogonal reflexions in the axes of x and y are

$$\begin{aligned} x' &= x \\ y' &= -y \end{aligned} \quad \text{and} \quad \begin{aligned} x' &= -x \\ y' &= y \end{aligned}$$

respectively. If by an orthogonal line reflexion a figure F is transformed into a figure F', then by the *same* transformation F' is transformed back to F. That is, a repeated orthogonal line reflexion gives identity. Now (5.1) is a special case of

$$\begin{aligned} x' &= ax + by + c_1 \\ y' &= bx - ay + c_2 \end{aligned} \qquad a^2 + b^2 = 1 \tag{5.2}$$

The transformation (5.2) is called a *symmetry*. As $a^2 + b^2 = 1$, we may put $a = \cos 2\theta$, $b = \sin 2\theta$; and therefore the necessary and sufficient condition for (5.2) being an orthogonal reflexion in a line is

$$c_1 = 2\omega \sin \theta, \quad c_2 = -2\omega \cos \theta,$$

where ω is any real number. Hence, as $a - 1 = -2 \sin^2\theta$,

$$\begin{vmatrix} a-1 & c_1 \\ b & c_2 \end{vmatrix} = 0$$

is a necessary condition for it. If $b \neq 0$, this condition is also sufficient

A vector $\alpha_k = (a_k, b_k)$ is transformed by the symmetry (5.2) into $\alpha'_k = (aa_k + bb_k, ba_k - ab_k)$. Hence the scalar product

$$\alpha'_1 . \alpha'_2 = (a_1 a_2 + b_1 b_2)(a^2 + b^2) = (a_1 a_2 + b_1 b_2) = \alpha_1 . \alpha_2$$

remains unaltered by every symmetry. We shall now prove the converse, namely the following theorem :

If by any linear transformation

$$x' = a_1 x + b_1 y + c_1$$

$$y' = a_2 x + b_2 y + c_2,$$

the scalar product of every pair of vectors remains unaltered, the transformation is either a rigid motion or a symmetry.

Proof. Let us consider the two orthogonal unit vectors

$$\alpha_1 = (1, 0), \quad \alpha_2 = (0, 1)$$

As in § 10.1, these are transformed into

$$\alpha'_1 = (a_1, a_2), \quad \alpha'_2 = (b_1, b_2)$$

As the scalar products are supposed to remain unaltered, we have

$$1 = \alpha_1 . \alpha_1 = \alpha'_1 . \alpha'_1 = a_1^2 + a_2^2$$

$$1 = \alpha_2 . \alpha_2 = \alpha'_2 . \alpha'_2 = b_1^2 + b_2^2$$

$$0 = \alpha_1 . \alpha_2 = \alpha'_1 . \alpha'_2 = a_1 b_1 + a_2 b_2$$

These three equations must be satisfied by the coefficients. Consider the two cases :

If $a_2 = 0$, then $a_1 = \pm 1$; hence $b_1 = 0$, and therefore $b_2 = \pm 1$. In this ase, the transformation is obviously either a rigid motion or a symmetry.

If $a_2 \neq 0$, put $b_1/a_2=\lambda$.

Then
$$b_1=\lambda a_2, \quad b_2=-\lambda a_1 ;$$
hence
$$1=b_1^2+b_2^2=\lambda^2$$

In this case, the transformation is a symmetry for $\lambda=1$ and is a rigid motion for $\lambda=-1$. Hence the theorem holds in every case.

It follows from above that $\lambda=\pm 1$ is equal to the determinant $a_1b_2-b_1a_2$ of the transformation. Therefore, *the rigid motions are those linear transformations which do not alter the scalar product of two vectors and which have a positive determinant ; the symmetries have the same property but with a negative determinant.* To give a geometrical interpretation of the determinant, let us consider the angle between two arbitrary vectors

$$\gamma=(g_1, g_2) \text{ and } \delta=(d_1, d_2).$$

Now
$$\cos(\gamma, \delta)=(\gamma.\delta)/[(\gamma.\gamma)\ (\delta.\delta)]^{\frac{1}{2}}$$

is a function of scalar products and will therefore be altered neither by rigid motions nor by symmetries ; and

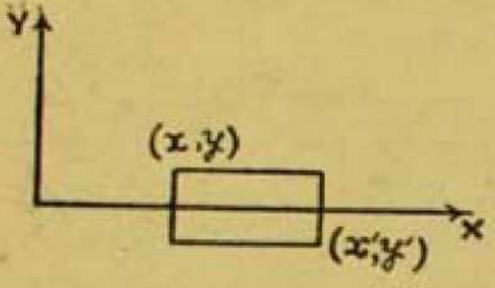
$$\sin(\gamma, \delta)=\begin{vmatrix} g_1 & g_2 \\ d_1 & d_2 \end{vmatrix} \Big/ [(\gamma.\gamma)\ (\delta.\delta)]^{\frac{1}{2}},$$

where the denominator is not altered by rigid motions or symmetries while the numerator takes the factor $\lambda=\pm 1$ only. Hence, *by rigid motions the angle remains unaltered, whereas by symmetries it is replaced by its negative.*

We may extend the notion of the product of two rigid motions as introduced in § 11.1, to the notion of the product of two linear transformations which are rigid motions or symmetries. The product is a linear transformation which does not alter the scalar products and is therefore a rigid motion or a symmetry. If both factors are rigid motions, or both are symmetries, the angle will not be altered by the product, and the product is therefore a rigid motion. If one factor is a rigid motion and the other is a symmetry, the sign of every angle will change, and the product is a symmetry. In general, the product of any number of rigid motions and of k symmetries is a rigid motion if k is even, and is a symmetry if k is odd.

If a symmetry is, in particular, the product of a translation and a reflexion in a line parallel to the line of translation, then the symmetry may be called a *paddle motion*.

As an illustration, consider a translation and an orthogonal reflexion in the x-axis given by



$$\bar{x}=x+c \qquad \text{and} \qquad x'=\bar{x}$$
$$\bar{y}=y \qquad\qquad\qquad y'=-\bar{y}$$

respectively. The product is the paddle motion

$$x'=x+c, \qquad y'=-y$$

It may be noted that the product of a translation and a reflexion in a line parallel to the line of translation is commutative.

15.1. Existence of fixed points. If the transformation (5.2) leaves any point fixed, we must have

$$(a-1)x+by+c_1=0$$
$$bx-(a+1)y+c_2=0$$

As the determinant of the coefficients

$$\begin{vmatrix} a-1 & b \\ b & -(a+1) \end{vmatrix} = 1-(a^2+b^2) = 0,$$

vanishes identically, there is either no fixed point or an infinity of fixed points. For an infinity of fixed points, we must have

$$\begin{vmatrix} b & c_1 \\ -(a+1) & c_2 \end{vmatrix} = \begin{vmatrix} a-1 & c_1 \\ b & c_2 \end{vmatrix} = 0$$

Put $\qquad a=\cos 2\theta, \qquad b=\sin 2\theta.$

So $\qquad a-1=-2\sin^2\theta, \quad a+1=2\cos^2\theta, \quad b=2\sin\theta\cos\theta$

Therefore $\quad 2\cos\theta\,(c_1\cos\theta+c_2\sin\theta)=0, \qquad 2\sin\theta\,(c_1\cos\theta+c_2\sin\theta)=0$

As $\sin\theta$ and $\cos\theta$ cannot both be zero, we must have

$$c_1\cos\theta+c_2\sin\theta=0, \quad \text{or} \quad c_1/c_2=-\sin\theta/\cos\theta$$

So, we may put

$$c_1=2\omega\sin\theta, \qquad c_2=-2\omega\cos\theta,$$

Therefore, in this case, (5.2) becomes (5.1) and the locus of fixed points is (5.1′), namely

$$x\sin\theta-y\cos\theta=\omega$$

If, however, there is no fixed point,

$$c_1\cos\theta+c_2\sin\theta=\sigma, \text{ where } \sigma\neq 0$$ ✓

So, we may put

$$c_1 = 2\rho \sin\theta + \sigma\cos\theta, \qquad c_2 = -2\rho\cos\theta + \sigma\sin\theta$$

The transformation (5.1) now reduces to

$$\begin{aligned} x' &= x\cos 2\theta + y\sin 2\theta + 2\rho\sin\theta + \sigma\cos\theta \\ y' &= x\sin 2\theta - y\cos 2\theta - 2\rho\cos\theta + \sigma\sin\theta \end{aligned} \qquad (5.3)$$

This transformation can be resolved into ✓

$$\begin{aligned} \bar{x} &= x + \sigma\cos\theta \\ \bar{y} &= y + \sigma\sin\theta \end{aligned} \quad \text{and} \quad \begin{aligned} x' &= \bar{x}\cos 2\theta + \bar{y}\sin 2\theta + 2\rho\sin\theta \\ y' &= \bar{x}\sin 2\theta - \bar{y}\cos 2\theta - 2\rho\cos\theta \end{aligned}$$

The first is a translation and the second is an orthogonal line reflexion, their product being commutative. Therefore, the transformation (5.3) is a paddle motion.

Thus, *a symmetry is either an orthogonal line reflexion or a paddle motion.*

16. Similarity. Consider the transformation

$$\begin{aligned} x' &= cx \\ y' &= cy \end{aligned} \qquad c \neq 0 \qquad (5.4)$$

The transformation (5.4) is called a *dilation*. It may be of the following kinds :

(*i*) If $c = 1$, the transformation is an *identity*.

(*ii*) If $c = -1$, the transformation is called a *reflexion in a point* (in the origin). In this case, if P' is the transform of a point P, the segment PP' is bisected at the origin.

(*iii*) If $c > 0$ but $\neq 1$, the transformation is called a *radial transformation* (from the origin).

(*iv*) If $c < 0$ but $\neq -1$, the transformation is a *product* of a radial transformation from the origin and reflexion in the origin.

We may look upon the transformations (*iii*) and (*iv*) as effecting a change in the unit. A plane figure is transformed into a similar figure. In particular, there is *stretching* or magnification if $|c| > 1$ and *shrinking* if $|c| < 1$.

If (5.4) is followed by a translation, we have a transformation of the form

$$\begin{aligned} x' &= cx + c_1 \\ y' &= cy + c_2 \end{aligned} \qquad c \neq 0 \qquad (5.5)$$

The transformation (5.5) is called a *homothetic transformation*. A homothetic transformation is

(*i*) a translation, if $c=1$,

(*ii*) a point reflexion (in the point $(-c_1, -c_2)$), if $c=-1$,

(*iii*) a *similitude*, if $c \neq \pm 1$ (c_1, c_2 may be both zero),

(*iv*) a radial transformation (from $(-c_1, -c_2)$), if $c > 0$ but $\neq 1$

and (*v*) a product of a radial transformation (from $(-c_1, -c_2)$) and a point reflexion, if $c < 0$ but $\neq -1$.

The product of (5.4) and a rigid motion is a transformation of the form

$$\begin{aligned} x' &= ax+by+c_1 \\ y' &= -bx+ay+c_2 \end{aligned} \qquad a^2+b^2=c^2, \tag{5.6}$$

The transformation (5.6) is called a *similarity*.

Rigid motion preserves both the shape and the size of a figure, whereas similarity preserves the shape, but not necessarily the size. As in the case of a rigid motion, there is a fixed point in a similarity unless it is a translation.

The following are some of the properties of (5.6) and (5.2). If α, β are two vectors and θ the angle between them, from one to the other, and if α', β', θ', are the transforms of α, β, θ then

(1) $| \alpha' | = | ca |$, by (5.6) ; $| \alpha' | = | \alpha |$, by (5.2)

(2) $\theta'=\theta$, by (5.6) ; $\theta'=-\theta$, by (5.2)

(3) The product of two similarities is a similarity. This is not true of symmetries.

(4) The inverse of a similarity is a similarity. This is true of a symmetry.

In (5.5) let $c \neq 1$, *i.e.*, let the homothetic transformation be not a translation. Then

$$(x_0, y_0) = \left(\frac{c_1}{1-c}, \ \frac{c_2}{1-c}\right)$$

is a fixed point P_0. Introducing new coordinates

$$\xi = x - x_0, \qquad \eta = y - y_0,$$

we get a standard form of the homothetic transformation as

$$\xi' = c\xi, \qquad \eta' = c\eta$$

The straight line joining an arbitrary pair of corresponding points $P=(\xi, \eta)$ and $P'=(\xi', \eta')$ passes through P_0, and $\overline{P_0P'} = c\overline{P_0P}$.

Corresponding straight lines remain parallel, and so the angle between them remains unaltered by any homothetic transformation. Any figure F will be transformed by (5.7) into a ***homothetic figure*** F', P_0 being the ***homothetic centre***. On the other hand, figures which are similar to F and in which the distances are equal to the corresponding distances in F multiplied by a constant factor $c > 0$, are all congruent. We can construct such a figure by the help of the transformation (5.7), or by

$$\xi' = -c\xi, \qquad \eta' = -c\eta,$$

by choosing the homothetic centre (x_0, y_0) in an arbitrary manner. Hence, every figure similar to F can be carried by a rigid motion in such a position that it becomes homothetic to F. Therefore, homothetic figures are also said to be ***similar and similarly situated***.

After rotating any figure homothetic to F through the homothetic centre P_0, we get a figure F' which is said to be ***directly similar to F***. If P and P' are corresponding points of F and F', $\sphericalangle PP_0P'$ is constant.

CHAPTER VI

THE CIRCLE

17. Power of a point with respect to a circle. A circle may be regarded as a special form of an ellipse. An ellipse was defined in § 12 as a nondegenerate conic whose equation can be put in the normal form (4.4), and in § 14 as the locus of a point the sum of whose focal distances is constant. We may also define an ellipse as follows: Let A, A' be two fixed points, P a variable point, M the foot of the perpendicular drawn from P to AA'. Then the locus of P such that $|PM|^2=k\,\overline{AM}.\overline{MA'}$, where k is a constant, is a conic. The conic is an ellipse or a hyperbola according as k is positive or negative and, in the limiting case when $k=0$, the conic is a parabola. There are here three definitions of a circle. In the first case, the ellipse is a circle if $a^2=b^2$; in the second case, a circle is the limiting form of an ellipse when its two foci coincide and in the third, an ellipse reduces to a circle when $k=1$. We shall, however, consider here the properties of the circle independently of any other second degree curve.

Let a straight line through a given point $P_0=(x_0, y_0)$ be given by the equations

$$x=x_0+\rho a$$

$$y=y_0+\rho b,$$

where ρ is a parameter. If θ is the angle between the positive x-axis and the vector (a, b),

$$a = |\sqrt{(a^2+b^2)}|\cos\theta, \quad b = |\sqrt{(a^2+b^2)}|\sin\theta$$

So, the above parametric equations of the straight line can be written as

$$\begin{aligned} x&=x_0+r\cos\theta \\ y&=y_0+r\sin\theta, \end{aligned} \tag{6.1}$$

where θ is a constant and r is a parameter.

Now, when r is regarded as a constant and θ as a parameter, the equations (6.1) give the parametric equations of a circle. Eliminating the parameter θ, we obtain the equation of a circle as

$$(x-x_0)^2+(y-y_0)^2-r^2=0. \tag{6.2}$$

The point P_0 is the *centre* and r the *radius* of the circle. The circle is, of course, without real trace or a null circle according as $r^2 \leq 0$.

Let the above equation be written as

$$x^2+y^2+l(x,\, y)=0, \tag{6.3}$$

where $l(x, y)$ is a linear function equal to $c_1x+c_2y+c_3$, say. Does the equation (6.3) represent a circle? If it does, it must be possible to reduce this equation to the form (6.2). In this case

$$c_1=-2x_0, \quad c_2=-2y_0, \quad c_3=x_0^2+y_0^2-r^2;$$

therefore

$$4r^2=c_1^2+c_2^2-4c_3.$$

Since $r^2>0$, the required condition is $c_1^2+c_2^2-4c_3>0$.

For the points P', P'' of intersections of any straight line given by

$$\left.\begin{aligned} x&=x_1+\rho p\\ y&=y_1+\rho q \end{aligned}\right. \qquad p^2+q^2=1,$$

through a given point $P_1=(x_1, y_1)$ and the circle (6.2), we must have

$$\rho^2-2B\rho+C=0,$$

where

$$B=p(x_0-x_1)+q(y_0-y_1),$$

$$C=(x_0-x_1)^2+(y_0-y_1)^2-r^2$$

Therefore, for the points of intersection, the scalar product must satisfy

$$\overline{P_1P'}.\ \overline{P_1P''}=(x_1-x_0)^2+(y_1-y_0)^2-r^2$$

This is a constant as long as the point (x_1, y_1) and the circle are given. The quantity on the left-hand side is called *the power of the point P_1 with respect to the circle.* The power is thus obtained by substituting the coordinates of P_1 for x, y in the expression on the left-hand side of the equation (6.2) of the circle. Evidently, the power is positive, zero or negative according as P_1 lies outside, on or inside the circle.

If the straight line P_1P' passes through the centre of the circle, then the power of P_1

$$=|\,P_1P_0\,|^2-r^2$$

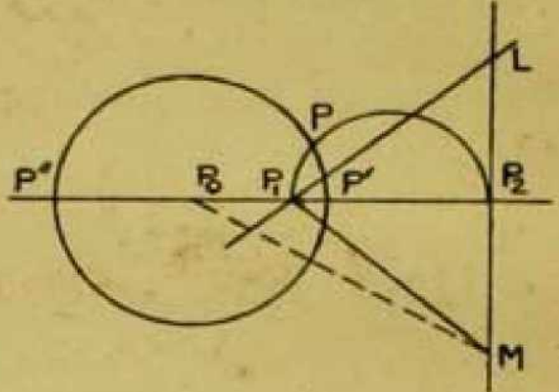
= square of the distance between P_1 and the point of contact of a tangent drawn from P_1 to the circle, if the point P_1 is outside the circle. The equation of the tangent at a point (ξ, η) of the circle being

$$(x-x_0)(\xi-x_0)+(y-y_0)(\eta-y_0)-r^2=0,$$

the tangent is evidently orthogonal to the straight line joining the centre of the circle and the point (ξ, η).

18. Pole, polar. As a particular case of the equation (4.12), the equation of the polar of a point $P_1=(x_1, y_1)$ with respect to the circle (6.2) (when P_1 is not the centre of the circle) is

$$(x-x_0)(x_1-x_0)+(y-y_0)(y_1-y_0)-r^2=0$$



This shows that *the polar of a point with respect to a circle is orthogonal to the line joining the point and the centre of the circle.*

As in § 13, let the straight line through P_1 and the centre P_0 of the circle meet the circle in P', P'' and the polar of P_1 in P_2. Then

$$(P_1P_2, P'P'')=-1$$

Hence, by (2.3),
$$\overline{P_0P_1}\cdot\overline{P_0P_2}=r^2 \tag{6.4}$$

Consider any circle passing through the points P_1, P_2 and let the two circles intersect one another in a point P.

Since, by (6.4),
$$\overline{P_0P_1}\cdot\overline{P_0P_2}=|P_0P|^2,$$
the power of P_0 with respect to the circle through P_1, P_2 is $|P_0P|^2$. Hence the straight line P_0P is tangent to this circle. Thus, *given the four harmonic points, the circle with the segment $P'P''$ (or P_1P_2) as a diameter cuts any circle through P_1, P_2 (or P', P'') orthogonally.*

Further, let P_1L, P_1M be two conjugate lines (§ 13) with respect to the circle having P_0 as centre, meeting the polar of P_1 in L, M. Then the triangle P_1LM is a *self-conjugate* triangle with respect to this circle. Therefore, the straight lines joining P_0 with the vertices P_1, L, M are orthogonal to the opposite sides of the triangle. Hence, P_0 is the orthocentre of the triangle P_1LM. Thus, *if a triangle is self-conjugate with respect to a circle, the centre of the circle is the orthocentre of the triangle.*

19. Coaxal system. Consider two circles with different centres

$$K_1\equiv(x-x_1)^2+(y-y_1)^2-r_1^2\equiv x^2+y^2+l_1(x, y)=0$$
$$K_2\equiv(x-x_2)^2+(y-y_2)^2-r_2^2\equiv x^2+y^2+l_2(x, y)=0$$

If the power of a point (x, y) with respect to the two circles be the same, we must have

$$l_1(x, y)-l_2(x, y)=0 \tag{6.5}$$

Since this is a linear equation, the locus of points having the same power with respect to both the circles is a straight line. This straight line is called the *radical axis* of the two circles. Let

$$K_3\equiv x^2+y^2+l_3(x, y)=0$$ ✓

be an arbitrary circle. If its centre does not lie on the straight line joining the centres of $K_1=0$, $K_2=0$, the radical axes of $K_1=0$, $K_3=0$ and $K_2=0$, $K_3=0$ are nonparallel and are given by (denoting the linear functions by the letters l_i only)

$$l_1-l_3=0 \quad \text{and} \quad l_2-l_3=0$$

respectively. This shows that *the radical axes of three circles whose centres are noncollinear, taken in pairs, meet in a point.*

On the other hand, if any two (and therefore, the three) radical axes of the three circles $K_1=0$, $K_2=0$, $K_3=0$, taken in pairs, coincide, we may write

$$l_1-l_3=\rho(l_1-l_2),$$

or $$l_3=(1-\rho)l_1+\rho l_2=\gamma l_1+\lambda l_2, \quad \text{say}$$

Therefore $$K_3=\gamma K_1+\lambda K_2, \quad \text{where } \gamma+\lambda=1$$

Accordingly, the equation of $K_3=0$ can be written as

$$\mu K_1+vK_2=0, \quad \mu+v\neq 0. \tag{6.6}$$

The totality of circles given by the equations (6.6) for arbitrary values of μ, v, satisfying $\mu+v\neq 0$, is said to form *a system of coaxal circles.* In a coaxal system the radical axis of each pair of circles is the same.

Since $\mu+v\neq 0$, we may divide the equation (6.6) by $\mu+v$ and write the equation as

$$\gamma K_1+\lambda K_2=0, \quad \text{where } \gamma+\lambda=1\ ;$$

that is, we make the co-efficients of x^2, y^2 unity. Then

$$\gamma K_1+\lambda K_2\equiv\{x-(\gamma x_1+\lambda x_2)\}^2+\{y-(\gamma y_1+\lambda y_2)\}^2-r_3^2$$

Thus, *the centres of a system of coaxal circles lie on a straight line.*

If $\mu+v=0$, $$\mu K_1+vK_2=\mu(K_1-K_2)\ ;$$

so, in this case $\mu K_1+vK_2=0$ is the radical axis (6.5) of the system of coaxal circles, unless the circles are concentric.

Further, $$K_1-K_2=2(x_2-x_1)x+2(y_2-y_1)y+\text{a constant},$$

where (x_1, y_1) and (x_2, y_2) are the centres of $K_1=0$, $K_2=0$.

This shows that *the radical axis is orthogonal to the line joining the centres.*

As an *application*, let AB, BC, CD, DA, be the sides of a quadrilateral and let the straight lines AB, CD meet in E ; BC, DA meet in F. The six points A, B, C, D, E, F are said to be the vertices of a complete quadrilateral of which AC, BD, EF are the three diagonals (the properties of such a figure will be discussed later in § 29.1).

Let S_1, S_2, S_3 be the three circles described on the segments AC, BD, EF respectively as diameters. Let O_1, O_2, O_3, O_4, be the orthocentres of the triangles EBC, ABF, ADE, CFD respectively (that is, the triangles formed by every three of the four sides of the quadrilateral). Also let EO_1, BO_1, CO_1, meet the opposite sides of the triangle EBC in E_1, B_1, C_1. Then, from similar right-angled triangles,

$$|O_1E|\,|O_1E_1| = |O_1B|\,|O_1B_1| = |O_1C|\,|O_1C_1|$$

But (C, C_1), (B, B_1), (E, E_1) are pairs of points on the circles S_1, S_2, S_3 respectively. Therefore, the powers of O with respect to the three circles are the same. Similarly the powers of each of the points O_2, O_3, O_4 with respect to the three circles are equal. Hence, the circles S_1, S_2, S_3 are coaxal, their radical axis containing the points O_1, O_2. O_3, O_4. Moreover, the middle points of the segments AC, BD, EF, which are the centres of these coaxal circles, must lie on a straight line. Hence, we may state the following theorem :

Imp *The circles described on the three diagonals of a complete quadrilateral as diameters are coaxal ; the middle points of these diagonals are collinear ; the orthocentre of the four triangles formed out of the four sides of the quadrilateral are also collinear ; the last two lines are respectively the line of centres and the radical axis of the coaxal circles.*

A circle coaxal with two given circles may also be defined as the locus of a point which moves so that the ratio of its powers with respect to the two given circles is constant. For, if $K_1=0$, $K_2=0$ are the two given circles and c is the constant ratio, then the locus is

$$K_1/K_2=c, \quad \text{or} \quad K_1-cK_2=0.$$

19.1. Types of coaxal system. Orthogonal system. Let the line of centres of a system of coaxal circles be taken as the x-axis and the radical axis as the y-axis. If two arbitrary circles of the system be

$$x^2+y^2+2\sigma_1x+c_1=0, \qquad x^2+y^2+2\sigma_2x+c_2=0,$$

their radical axis is

$$2(\sigma_1-\sigma_2)x+(c_1-c_2)=0, \quad \therefore \quad c_1-c_2=0$$

Put $\quad \sigma_2-\sigma_1=\sigma \quad$ and $\quad c_1=c_2=c$

Then the equation of the coaxal system can be written as

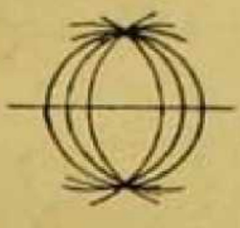
$$x^2+y^2+2\sigma x+c=0$$

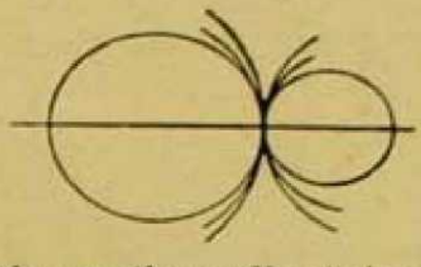
or

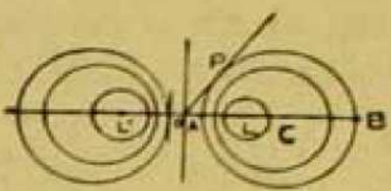
$$(x+\sigma)^2+y^2-(\sigma^2-c)=0, \tag{6.7}$$

where c is a constant and σ a parameter. The centre is the point $(-\sigma, 0)$ and the square of the radius is σ^2-c. Three different cases arise :

(1) $c < 0$. In this case $\sigma^2 - c > \sigma^2$. Therefore, every circle of the system cuts the radical axis in two fixed points. The system is said to be *elliptic*.







(2) $c = 0$. In this case the radius is $| \sigma |$. Therefore, all circles of the system touch the radical axis at the point of intersection of the radical axis and the line of centres. The system is said to be *parabolic*.

(3) $c > 0$. In this case $\sigma^2 - c < \sigma^2$. Therefore, no circle of the system can cut the radical axis (in real points). The system is said to be *hyperbolic*. We notice here that if $\sqrt{c} = \pm\sigma$, we obtain two circles of the system of zero radius, or two point-circles. These point-circles are called *the limiting points of the system*.

Let L, L' be the two limiting points, O the point of intersection of the radical axis and the line of centres. Let C be the centre of an arbitrary circle of the system of radius r meeting the line of centres in A, B and P be the point of contact of the tangent drawn from O to this circle, as in the figure.

Then,

$$| OL' |^2 = | OL |^2 = | OP |^2 = | OA | \; | OB | = \overline{OA}.\overline{OB}$$

Therefore, applying (2.1'), (2.3) and remembering that C is the middle point of AB, we get

$$(LL', AB) = -1, \quad \overline{CL}.\,\overline{CL'} = r^2 \tag{6.8}$$

An arbitrary point of the radical axis of the system of coaxal circles (6.7) has the coordinates $(0, -\rho)$. The power of this point with respect to to the system is

$$\sigma^2 + \rho^2 - (\sigma^2 - c) = \rho^2 + c$$

Therefore, the circle with centre $(0, -\rho)$ and radius $| \sqrt{(\rho^2 + c)} |$ will cut the circles of the system orthogonally. The equation of the circle is

$$x^2 + (y + \rho)^2 - (\rho^2 + c) = 0, \tag{6.9}$$

where ρ is a parameter. Comparing this equation with (6.7), we see that (6.9) represents a second system of coaxal circles. The two systems are *orthogonal* to one another.

The radical axis and the line of centres of (6.7) are respectively the line of centres and the radical axis of (6.9). Also, corresponding to the system (6.7) being elliptic, parabolic, hyperbolic, the system (6.9) is hyperbolic, parabolic, elliptic. This follows from the fact that the quantity c has different signs prefixed to it in the two equations.

20. Centres of similitude of two circles. Consider a transformation (5.5) of similitude

$$\begin{aligned} x' &= cx + c_1 \\ y' &= cy + c_2 \end{aligned} \qquad c \neq 0, \ \pm 1$$

The fixed point of the transformation is given by

$$x = c_1/(1-c), \quad y = c_2/(1-c)$$

The fixed point is called the *centre of similitude* and c is called the *ratio of similitude*. A straight line passing through the centre of similitude is transformed into itself, and any other straight line is transformed into a parallel straight line.

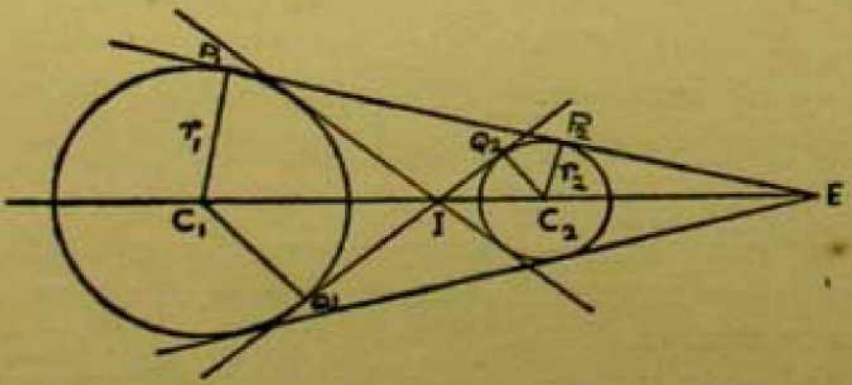
Suppose it is required to transform the circle $(x-x_1)^2+(y-y_1)^2-r_1^2=0$ with centre C_1 into another circle $(x-x_2)^2+(y-y_2)^2-r_2^2=0$ with centre C_2 by transformations of similitude. Then there are two such transformations :

$$x - x_1 = \pm \frac{r_1}{r_2}(x' - x_2)$$

$$y - y_1 = \pm \frac{r_1}{r_2}(y' - y_2),$$

according as we take both the upper or both the lower signs. The transformations have respectively r_2/r_1 and $-r_2/r_1$ as the ratios of similitude. The two centres of similitude corresponding to the two transformations are the points

$$E = \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \ \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right) \text{ and } I = \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \ \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$



respectively. From the coordinates of the points it is evident that

C_1, C_2, E, I are collinear, and that $(C_1C_2, EI) = -1$. The coordinates of the vectors $\overline{C_1E}$ and $\overline{C_2E}$ are

$$\left(\frac{r_1(x_2-x_1)}{r_1-r_2}, \frac{r_1(y_2-y_1)}{r_1-r_2}\right) \text{ and } \left(\frac{r_2(x_2-x_1)}{r_1-r_2}, \frac{r_2(y_2-y_1)}{r_1-r_2}\right) \text{ respectively.}$$

Therefore
$$\frac{\overline{C_1E}}{\overline{C_2E}} = \frac{r_1}{r_2}, \quad \frac{\overline{C_1I}}{\overline{C_2I}} = -\frac{r_1}{r_2}.$$

Hence, *the centres of similitude E, I are respectively the points of intersections of the direct common tangents and of the inverse common tangents to the two circles.* The points E and I are respectively called the *external* and the *internal* centres of similitude. The circle on the segment EI as a diameter is called *the circle of similitude.*

It is geometrically evident that when the two circles are mutually external, we obtain the four real and distinct common tangents. When the circles touch, one pair of common tangents coincide and one centre of similitude is the point of contact of the two circles. When the circles have equal radii, $c = \pm 1$; for the homothetic transformation with $c = +1$, there is no centre of similitude; if $c = -1$, the external centre of similitude does not exist. When the circles are concentric, we cannot have any centre of similitude.

Let P_1, P_2 be the points of contact of a common tangent drawn from E to the two given circles of radii r_1, r_2 respectively; similarly let Q_1, Q_2 be the points of contact of a common tangent drawn from I to these two circles respectively, as in the figure. Then

$$\frac{|EP_1|^2}{|EP_2|^2} = \frac{|IQ_1|^2}{|IQ_2|^2} = \frac{r_1^2}{r_2^2}$$

Therefore, by what has been said at the end of § 19, a circle drawn through E coaxal with the two given circles must pass through I and must have its centre collinear with E, I. Thus, *the circle of similitude is coaxal with the two given circles.*

Finally, consider three circles with centres (x_1, y_1), (x_2, y_2), (x_3, y_3) and radii r_1, r_2, r_3. Each pair of circles gives two centres of similitude. Therefore, there are six centres of similitude, three external and three internal. We can write down the coordinates of these six points, the coordinates of two of the points (namely, E, I) having been given above. Since the determinant

$$\begin{vmatrix} r_1x_2-r_2x_1 & r_1y_2-r_2y_1 & r_1-r_2 \\ r_2x_3-r_3x_2 & r_2y_3-r_3y_2 & r_2-r_3 \\ r_3x_1-r_1x_3 & r_3y_1-r_1y_3 & r_3-r_1 \end{vmatrix}$$

vanishes (as may be seen by multiplying the rows by r_3, r_1, r_2 respectively and adding), it follows that the three external centres of similitude are collinear. Similarly, the external centre of similitude of one pair of circles is collinear with the two internal centres of similitude of the remaining pairs of circles. Thus, *the six centres of similitude lie by threes on four straight lines*. These straight lines are known as the *axes of similitude* of the three circles.

21. Inversion. Let O be a fixed point and k^2 a given constant. If on the straight line joining O to a point P, a point P' is taken such that $OP.OP'=k^2$, then P' is called the *inverse* of P with respect to O and O the *centre of inversion*. We shall suppose that k^2 is positive ; we shall then have a circle with centre O and radius $|k|$. This circle is called the *circle of inversion* and $|k|$ the *radius of inversion*. Denoting this circle by (O), we say that P' is the inverse of P *with respect to* the circle (O) or *in the* circle (O). The relation between P and P' is symmetrical, so that P is also the inverse of P'. The points P and P' lie on the same side of O, one inside and the other outside (O), and each point of (O) is transformed by inversion into itself. Any circle concentric with (O) is transformed into a concentric circle.

Let the centre of inversion O be taken as the origin of the coordinate system. If P, P' have the coordinates (x, y), (x', y'),

then
$$\frac{x'}{x}=\frac{y'}{y}=\frac{\overline{OP'}}{\overline{OP}}=\frac{k^2}{|OP|^2}=\frac{|OP'|^2}{k^2}$$

Therefore
$$x'=\frac{k^2x}{x^2+y^2}, \qquad y=\frac{k^2y}{x^2+y^2}$$

$$\tag{6.10}$$

and so
$$x=\frac{k^2x'}{x'^2+y'^2}, \qquad y=\frac{k^2y'}{x'^2+y'^2}$$

Thus to every point, except the centre of inversion O, there corresponds an inverse point.

The inverse of straight lines and circles. We know that

$$d(x^2+y^2)+2ax+2by+c=0, \quad a^2+b^2>dc, \tag{6.11}$$

represents a circle if $d \neq 0$, and a straight line if $d=0$. In particular, the curve passes through the origin if $c=0$. The quantities a, b, c, d, may be considered as the *coordinates of the curve*; these coordinates may have a common arbitrary factor other than zero. Multiplying (6.11) by $k^2/(x^2+y^2)$, we get from (6.10)

$$ck^{-2}(x'^2+y'^2)+2ax'+2by'+dk^2=0 \tag{6.11'}$$

Thus, the curve (6.11) is transformed into a curve (6.11′) of the same type, the coordinates a, b, c, d, being transformed into a, b, dk^2, ck^{-2}. Hence, circles and straight lines are transformed by inversion into circles or straight lines, the image being a straight line if and only if the original curve passes through the centre of inversion.

Conformal properties of inversion.

A point transformation is said to be a *conformal transformation* if it preserves the magnitude of every angle; it is said to be *directly* conformal if the sense of the angle is also preserved, and *inversely* conformal if the sense is reversed. For example, rigid motion is directly conformal and symmetry is inversely conformal, as seen at the end of the last chapter.

Let (ξ, η) be a point of (6.11). If $d \neq 0$, the equation

$$(d\xi+a)x+(d\eta+b)y+a\xi+b\eta+c=0 \tag{6.12}$$

represents the tangent to the circle (6.11); if $d=0$, the equations (6.11) and (6.12) represent the same straight line. To avoid an unnecessary distinction of different cases, we shall consider every straight line as its own "tangent" at each of its points. It is therefore admissible to *define the angle between two curves* of the type (6.11), as the angle between the corresponding tangents (6.12); of course, this definition is quite natural, because if we consider a curve to be generated by a moving point, its velocity has the direction of the tangent. The angles between two circles intersecting in two different points form two opposite angles. Thus, the cosine, and not the sine, of the angle is uniquely determined for two circles. For calculating the angle, we therefore expect that the sine may be expressed by the coordinates of the curves and those of the point of intersection.

Let two curves of the type (6.11) having coordinates

$$(a_1, b_1, c_1, d_1) \text{ and } (a_2, b_2, c_2, d_2)$$

intersect in (ξ, η). Then, by (6.11) and (6.12),

$$d_1(\xi^2+\eta^2)+2a_1\xi+2b_1\eta+c_1=0$$

$$d_2(\xi^2+\eta^2)+2a_2\xi+2b_2\eta+c_2=0$$

and the two tangents are

$$(d_1\xi+a_1)x+(d_1\eta+b_1)y=\text{const.}$$

$$(d_2\xi+a_2)x+(d_2\eta+b_2)y=\text{const.}$$

For the angle ϕ between the tangents, we have

$$\cos\phi=A/B, \qquad \sin\phi=C/B, \tag{6.13}$$

where
$$A=(d_1\xi+a_1)(d_2\xi+a_2)+(d_1\eta+b_1)(d_2\eta+b_2)$$

$$B=[\{(d_1\xi+a_1)^2+(d_1\eta+b_1)^2\}\,\{(d_2\xi+a_2)^2+(d_2\eta+b_2)^2\}]^{\frac{1}{2}}$$

$$C=\begin{vmatrix} d_1\xi+a_1 & d_1\eta+b_1 \\ d_2\xi+a_2 & d_2\eta+b_2 \end{vmatrix}$$

The quantities A, B, C may be expressed as follows :

$$A=\tfrac{1}{2}d_2[d_1(\xi^2+\eta^2)+2a_1\xi+2b_1\eta]+\tfrac{1}{2}d_1[d_2(\xi^2+\eta^2)+2a_2\xi+2b_2\eta]+a_1a_2+b_1b_2$$

$$=a_1a_2+b_1b_2-\tfrac{1}{2}(c_1d_2+d_1c_2)$$

$$B^2=(a_1^2+b_1^2-c_1d_1)\,(a_2^2+b_2^2-c_2d_2)$$

$$C=\xi\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}+\eta\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}+\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

The last formula can be written in another form. For this purpose, we express d_1 and d_2 in terms of $\xi, \eta, a_1, a_2, b_1, b_2, c_1, c_2$ and obtain

$$\xi\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}=\frac{-\xi}{\xi^2+\eta^2}\begin{vmatrix} 2a_1\xi+2b_1\eta+c_1 & b_1 \\ 2a_2\xi+2b_2\eta+c_2 & b_2 \end{vmatrix}$$

$$=\frac{-\xi}{\xi^2+\eta^2}\left[2\xi\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}+\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}\right]$$

Apply the same method to the second determinant in C. Then

$$-C=\frac{\xi}{\xi^2+\eta^2}\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}+\frac{\eta}{\xi^2+\eta^2}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}+\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

To get the angle between the inverse curves, we have to use (6.10), (6.11) and replace in the formula (6.13) for $\cos\phi$ and $\sin\phi$ the quantities

$$\xi, \quad \eta, \quad a_1, \; b_1, \; c_1, \quad d_1, \quad a_2, \; b_2, \; c_2, \; d_2$$

respectively by $\dfrac{k^2\xi}{\xi^2+\eta^2}$, $\dfrac{k^2\eta}{\xi^2+\eta^2}$, a_1, b_1, k^2d_1, $k^{-2}c_1$, a_2, b_2, k^2d_2, $k^{-2}c_2$

This transformation does not alter A and B; but C in its first representation will be transformed to $-C$ in its second representation. Hence, the angle formed by the inverse curves is equal to $-\phi$. The transformation of the plane by inversion is therefore conformal, because the absolute value of the angle is not altered; and, as the sign is changed, the transformation is, in particular, inversely conformal.

Applications. (1) Every circle C intersecting the circle of inversion (O) at P and Q is inverted into a circle C' intersecting (O) at P and Q, the angle between the tangents to C and C' at P (and at Q) being bisected by the tangent of (O) at P (and at Q).

Every point of (O) is a fixed point of the inversion; thus, the proposition is a direct consequence of the fact that the inversion is inversely conformal.

If P is different from Q, the circle C' is uniquely given by P, Q and the two tangents. Hence:

(1′) *Every circle orthogonal to the circle of inversion is inverted into itself.* (The reader may give an alternative proof based directly on the definition of inversion.)

(2) *If an inversion carries a circle C and two points P, Q into a circle or a straight line C' and two points P', Q' and if P, Q are inverse points with respect to C, then P', Q' are inverse points with respect to C'.*

Since P, Q are inverse points with respect to C, any circle S passing through P, Q is orthogonal to C. The circle S is inverted into a circle S' passing through P', Q'. Since inversion is conformal, C' and S' are orthogonal. But as S is any circle, so by (1) above, P', Q' are inverse points with respect to C'.

(3) *The limiting points of a system of coaxal circles are inverse points with respect to every circle of the system.* This follows from (6.8).

22. Polar reciprocation with respect to a circle. In § 13, a correspondence between the points and the straight lines of the plane has been established by the help of a nondegenerate conic. In this article we shall consider, in a little more detailed manner, the case where this conic is a circle which, for simplicity, is supposed to be given by

$$x^2+y^2-1=0 \tag{6.14}$$

The polar of any point (ξ, η) with respect to (6.14) is the straight line $\xi x+\eta y-1=0$. By what we have said in the last article, the coordinates of the polar are therefore proportional to $\xi, \eta, -1$. Conversely, the pole of any straight line $ux+vy+w=0$ is the point with coordinates $(-u/w, -v/w)$. Hence, every point of the plane, except the centre $(0, 0)$ of the circle (6.14), has a polar and every straight line not passing through the centre has a pole. This result agrees with the ideas of § 13. As the tangent to any circle

$$(x-a)^2+(y-b)^2-r^2=0 \tag{6.15}$$

at a point (x_1, y_1) satisfies the equation

$$(x_1-a)x+(y_1-b)y-(ax_1+by_1+r^2-a^2-b^2)=0$$

it follows that the pole, with respect to (6.14), of this tangent has the coordinates (ξ, η) given by

$$\xi=(x_1-a)/(ax_1+by_1+r^2-a^2-b^2)$$

$$\eta=(y_1-b)/(ax_1+by_1+r^2-a^2-b^2)$$

Hence $$1-a\xi-b\eta=r^2/(ax_1+by_1+r^2-a^2-b^2)$$

Therefore $$x-a=\xi r^2/(1-a\xi-b\eta), \qquad y-b=\eta r^2/(1-a\xi-b\eta)$$

By putting these values in (6.15), the equation of the locus of the pole comes out (writing x, y for ξ, η) as

$$(r^2-a^2)\,x^2+(r^2-b^2)\,y^2-2abxy+2ax+2by-1=0 \tag{6.16}$$

The locus is therefore a conic. It is of the elliptic, the parabolic or the hyperbolic type according as

$$(r^2-a^2)(r^2-b^2)-a^2b^2 \gtreqless 0, \quad \text{or} \quad r^2 \gtreqless a^2+b^2$$

Thus we get the following result :

The locus (6.16) *of the poles, with respect to the circle* (6.14), *of the tangents to any circle* (6.15) *is a conic which is elliptic, parabolic or hyperbolic according as the centre of* (6.14) *is inside, on or outside the circle* (6.15).

To discuss the properties of the conic (6.16), it is advantageous to choose the coordinate system in such a manner that the join of the centres of (6.14) and (6.15) is the x-axis. That means $b=0$. Thus (6.16) gets the simpler form

$$(r^2-a^2)x^2+r^2y^2+2ax-1=0,$$

Specially in the parabolic case $r^2=a^2$,

$$y^2+\frac{2}{a}\left(x-\frac{1}{a}\right)=0$$

and in the other cases,

$$\left(\frac{r^2-a^2}{r}\right)^2\left(x+\frac{a}{r^2-a^2}\right)^2+y^2(r^2-a^2)=1$$

Thus, the conic is nondegenerate ; its main axis is the join of the centres of the circles. If the two centres coincide, the conic is a concentric circle with radius $1/r$. In every other case, the centre of (6.14) is a focus of (6.16). On the other hand, every parabola, ellipse or hyperbola, of which the origin is the focus and x-axis is the main axis, can be expressed by the above formulae and is therefore the locus of the poles of the tangents of a suitably chosen circle.

As before, let $P_1=(x_1, y_1)$ be a point of (6.15) and (ξ, η) a point of (6.16). When the point P_1 slides on the circle (6.15), the coordinates

$$x_1=a+r\cos\theta, \quad y_1=b+r\sin\theta$$

are continuous functions of θ. As long as

$$ax_1+by_1+r^2-a^2-b^2\neq 0,$$

the coordinates ξ and η are continuous functions of x_1 and y_1 and are therefore continuous functions of θ. When

$$r^2=a^2+b^2,$$

the circle (6.15) passes through the centre O of the circle (6.14) ; the tangent at this point has therefore no pole. There are zero, one or two tangents passing through O according as the circle (6.15) includes, passes through or excludes the point O. This consideration agrees with the previously obtained result that the locus of the pole is a closed curve (ellipse), an open curve with one branch (parabola), or an open curve with two branches (hyperbola) corresponding to these three cases.

Let A and B be two points of the circle (6.15), p_1 and p_2 be the tangents at these points, and P_1 and P_2 the poles with respect to (6.14) of these tangents ; P_1 and P_2 are therefore points of the curve (6.16). The line P_1P_2 is the polar of the point P in which p_1 and p_2 intersect. We suppose that no tangent at any point of the arc AB of (6.15) passes through O, the centre of (6.14). If B approaches A along this arc, P approaches A along p_1. As the coordinates of P_2 are continuous functions of θ, the point P_2 approaches P_1 and the line P_1P_2 approaches the tangent to the conic (6.16) at P_1. Let

$$ux+vy+w=0$$

be the equation of P_1P_2 ; then its pole P has the coordinates $(-u/w, -v/w)$. And as P approaches A, the quotients u/w and v/w

are continuously altered; for the polar of A with respect to (6.14), they take therefore the limit of the values that they have for the line P_1P_2. Hence, the polar of A is the tangent of the conic (6.16) at P_1.

Thus, there exists a *reciprocal* connection between the curves (6.15) and (6.16); the polar of every point of one of these curves is the tangent at the corresponding point of the other curve.

Applications. (1) *A system of nonintersecting coaxal circles can be reciprocated into confocal conics.*

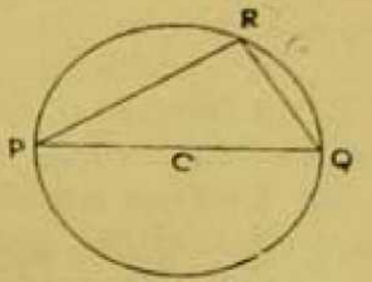
The asymptotes of a hyperbola (4.5) are given by (4.3). We *define* the asymptotes of an ellipse (4.4) as the pair of intersecting straight lines without real trace (*i.e.*, the null ellipse) given by (4.2). Both pairs of asymptotes satisfy the analytical condition of being tangents to the respective curves through their centres (§ 13).

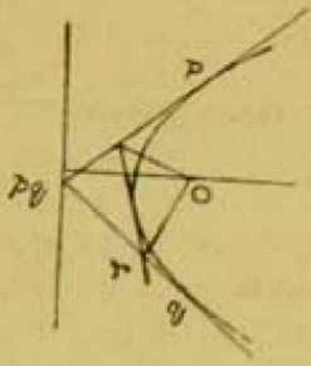
Let L, L' be the limiting points of a system of coaxal circles, and let us reciprocate the circles with respect to a circle (L) having one of the limiting points L as centre. Then the circles will reciprocate into non-degenerate conics having a common focus at L. If these conics are confocal, they must have also a common centre. Let an arbitrary circle C of the system reciprocate into a conic S. Since the asymptotes of S are tangents to S, the poles of the asymptotes with respect to (L) must lie on C and also on the perpendiculars drawn through L to the asymptotes. Therefore, to the centre of S (*i.e.*, the point of intersection of the asymptotes) corresponds the straight line joining the points of contact of tangents from L to C, namely, the polar of L with respect to C. Thus, all the conics will have the same centre if the polar of L with respect to all the circles of the system be the same. But this is true, because the polar of L with respect to any circle of the system is the straight line through L' perpendicular to the line of centres.

(2) Reciprocate with respect to a circle the theorem that the angle in a semi-circle is a right angle.

Reciprocation of properties involving angle is based on the obvious theorem that one of the angles between two straight lines is equal to the angle which their poles, with respect to a circle, subtend at the centre of the circle.

Let C be the centre of a circle (C) of which PQ is a diameter subtending a right angle at a point R of the circle. If (O) be a circle, the polar reciprocal of (C) with respect to (O) is a conic of which O is a focus and the polar of C is the corresponding directrix. To the points P, Q, R, correspond the tangents p, q, r to the conic. Let pq, qr, rp denote the points of intersections of these tangents. To the lines PQ, QR, RP, correspond the points *pq*, *qr*, *rp* respectivly, the point *pq* lying on the directrix, and the points *qr*, *rp* subtending a right angle at the focus.





Thus, the reciprocal theorem is that *the intercept on any tangent to a conic between two other tangents which intersect in a point on a directrix subtends a right angle at the corresponding focus.*

CHAPTER VII

AFFINITY

23. Affine transformations. Let the points A, B, ... of the plane be put in correspondence with the points A', $B,'$ . . . . of the plane in such a manner that to every (original) point A corresponds one and only one point, its "image", A'; and that every point of the plane is also the image of one and only one point of the plane. Every correspondence or representation of this kind is said to be a *transformation of the plane*. Rigid motions, symmetries, similarities are instances of such transformations. In general, every permutation of the points of the plane is a transformation. We shall consider here a special class of transformations which includes rigid motions, symmetries and similarities as special cases. In what follows, the image of any point will be denoted by affixing a dash.

Suppose that a transformation is such that to every vector there corresponds a vector. That means :

If $$\overline{A_1A_2}=\overline{B_1B_2}= \; . \; . \; . \; . =\alpha,$$

then $$\overline{A'_1A'_2}=\overline{B'_1B'_2}= \; . \; . \; . \; . =\alpha' ;$$

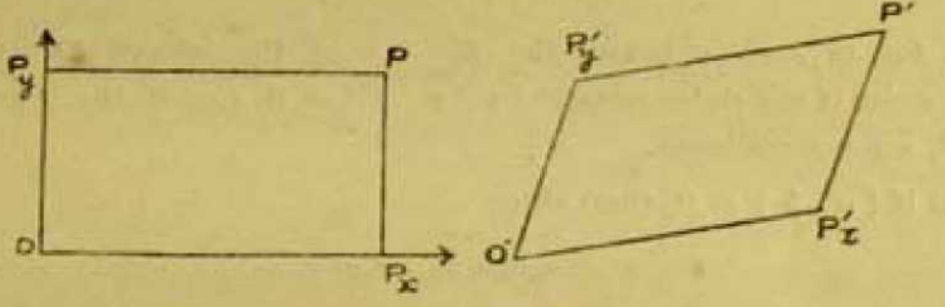
that is to say, different pairs of points defining the same vector α are transformed into pairs of points defining the same vector α'. It follows that if $\overline{A_1B_1}=\beta$, then $\overline{A'_1B'_1}=\beta'$, and the vector $\overline{A_1B_2}=\alpha+\beta$ is transformed into $\overline{A'_1B'_2}=\alpha'+\beta$. This formula holds for arbitrary vectors α and β. Hence, *if by a transformation of the plane vectors are transformed into vectors, the sum of two vectors is transformed into the sum of the corresponding vectors.*

From $\alpha=\beta$ it follows that 2α is transformed into $2\alpha'$. The reader may prove that in consequence of these suppositions, $r\alpha$ is transformed into $r\alpha'$, where r is any *rational* number. We shall *assume* further that for every *real* number λ, the vector $\lambda\alpha$ is transformed into $\lambda\alpha'$. Every transformation satisfying these conditions is called an *affine transformation* or an *affinity*.

Every vector of the plane is the affine image of one and only one vector. Since parallel vectors are obtained by multiplying a vector by real numbers, it follows that parallel vectors are transformed by the affinity into parallel vectors. On the other hand, if $\beta'=\lambda\alpha'$, then β' is the affine image of the vector $\beta=\lambda\alpha$. Hence, parallel vectors are also the images of parallel vectors and nonparallel vectors are transformed into nonparallel vectors. Considering these facts, we realise that the *inverse* of any affine transformation satisfies the conditions which are characteristic of an affine transformation and is therefore an affine transformation too. Moreover, two affinities, effected one after the other, furnish a transformation of the plane by which vectors are transformed into vectors and the product of a real number and a vector is transformed into the product of the same number and the corresponding vector. Hence the result is an affinity.

Analytical expression of affinities.

Consider a coordinate system of axes with O as the origin and let the points P, P_x, P_y have the coordinates (x, y), $(x, 0)$, $(0, y)$ respectively. Let the points O, P, P_x, P_y and the two unit vectors α_1, α_2 in the directions of $\overline{OP_x}$, $\overline{OP_y}$ be transformed by an affinity into O', P', P'_x, P'_y, α'_1, α'_2 respectively. Finally, let the coordinates of O', P', α'_1, α'_2 be (a_0, b_0), (x', y'), (a_1, b_1), (a_2, b_2) respectively.



Then $$\overline{O'P'_x} = x\,\alpha'_1, \qquad \overline{O'P'_y} = y\,\alpha'_2$$

Therefore $$\overline{O'P'} = x\alpha'_1 + y\alpha'_2$$

But the coordinates of $\overline{O'P'}$ are $(x' - a_0, y' - b_0)$. Hence

$$\begin{aligned} x' &= a_1 x + a_2 y + a_0 \\ y' &= b_1 x + b_2 y + b_0 \end{aligned} \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0 \tag{7.1}$$

The determinant of the coefficients is not zero, because the vectors α_1, α_2 are not parallel. This condition ensures that the transformation has its inverse. Thus, an affinity is given by (7.1). Obviously, an affinity transforms a straight line into a straight line.

For an affinity given by the linear transformation (7.1), the transformation of the vectors is given by the corresponding homogeneous transformation. Thus, if the vector $(x_2 - x_1, y_2 - y_1)$ is transformed into $(x'_2 - x'_1, y'_2 - y'_1)$, then

$$\begin{aligned} x'_2 - x'_1 &= a_1(x_2 - x_1) + a_2(y_2 - y_1) \\ y'_2 - y'_1 &= b_1(x_2 - x_1) + b_2(y_2 - y_1) \end{aligned}$$

We shall hereafter use shorter notations for determinants whenever there is no occasion for ambiguity ; for example,

$$| a \quad b | \text{ or } | a_1 \quad b_2 | \text{ will stand for } \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$| x \quad y \quad 1 | \text{ or } | x_1 \quad y_2 \quad 1 | \text{ will stand for } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now, there are six constants in (7.1) and so six conditions may be expected to determine them. Let three points $P_i=(x_i, y_i)$ be transformed by (7.1) into the three points $P'_i=(x'_i, y'_i)$, $i=1, 2, 3$. Then

$$x'_i=a_1x_i+a_2y_i+a_0$$
$$y'_i=b_1x_i+b_2y_i+b_0$$

Solutions for (a_1, a_2, a_0) and (b_1, b_2, b_0) of the above two systems of equations exist if the determinant $| x \quad y \quad 1 | \neq 0$, *i.e.*, if the three points P_1, P_2, P_3 are noncollinear.

Again if $| a \quad b | = 0$, then since

$$x'_i-a_0=a_1x_i+a_2y_i$$
$$y'_i-b_0=b_1x_i+b_2y_i,$$

we must have

$$\rho(x'_i-a_0)+\sigma(y'_i-b_0)=0, \quad \rho, \sigma \neq 0$$

Or, the three equations

$$\rho x'_i+\sigma y'_i-(\rho a_0+\sigma b_0)=0, \quad i=1, 2, 3,$$

in the unknowns ρ, σ, must hold simultaneously. Hence

$$| x' \quad y' \quad 1 | = 0 ;$$

and therefore the three points P_1, P_2, P_3 are collinear.

Thus, *an affinity is uniquely determined when three given noncollinear points are transformed by it into three other given noncollinear points.*

For example, the affinity which carries three noncollinear points (x_i, y_i), into the three points (0,0), (1,0), (0,1) respectively is given by

$$x'=\begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} \div \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \quad y'=\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \div \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

As before, let the three points (x_i, y_i) be transformed by (7.1) into the three points (x'_i, y'_i) and Δ, Δ' be the areas of triangles formed by the two triads of points. Then

$$\begin{aligned} 2\Delta' &= |\, x' \;\; y' \;\; 1 \,| \\ &= |\, a_1x+a_2y+a_0 \;\; b_1x+b_2y+b_0 \;\; 1 \,| \\ &= |\, a_1x+a_2y \;\; b_1x+b_2y \;\; 1 \,| \\ &= a_1b_2 \,|\, x \; y \; 1 \,| + a_2b_1 \,|\, y \; x \; 1 \,| = |\, a \; b \,| \; |\, x \; y \; 1 \,| \end{aligned}$$

Therefore $\qquad \Delta' = |\, a \; b \,| \, \Delta$

Two figures are said to be *affine* (or "equivalent" in the sense of affinity) if there exists an affinity transforming one into the other. Hence, *all triangles are affine*. The length of a vector and the angles between vectors will, in general, be altered by affinity. In § 15 we have obtained a result which can be expressed in the following manner :

Rigid motions and symmetries are those affinities by which the scalar product of two vectors is not altered.

23.1 Particular cases. Consider the transformation

$$\begin{aligned} x' &= x \\ y' &= by \end{aligned} \qquad b \neq 0 \tag{7.2}$$

By (7.2), every point of the x-axis is transformed into itself ; any line parallel to the x-axis is transformed into a parallel line and any parallel to the y-axis is transfomed into itself. In general, any line intersecting the x-axis in a point is transformed into a line passing through the same point on the x-axis.

To understand this transformation in an intuitive manner when b is positive and greater then 1, imagine the plane to be stretched, like an elastic membrane, directly away from the axis of x on both sides of it, so that each point is carried along a line parallel to the axis of y. A point (x, y) will be carried to a point (x, by). Therefore, a figure will be *elongated* away from the axis of x. When b is positive and less than 1, we have *compression* towards the axis of x. Similarly, we might have elongation away from or compression towards the axis of y. When b is positive, the transformation (7.2) is called a *simple strain*. When b is negative, it is the product of a simple strain and an orthogonal reflexion in the x-axis.

Consider the transformation

$$\begin{aligned} x' &= ax \\ y' &= by \end{aligned} \qquad ab \neq 0 \tag{7.3}$$

By (7.3), the origin is transformed into itself and any line parallel to a coordinate axis is transformed into a parallel line. The transformation may be factorised into

$$\begin{aligned}\bar{x}&=x\\ \bar{y}&=by\end{aligned} \quad \text{and} \quad \begin{aligned}x'&=a\bar{x}\\ y'&=\bar{y}\end{aligned}$$

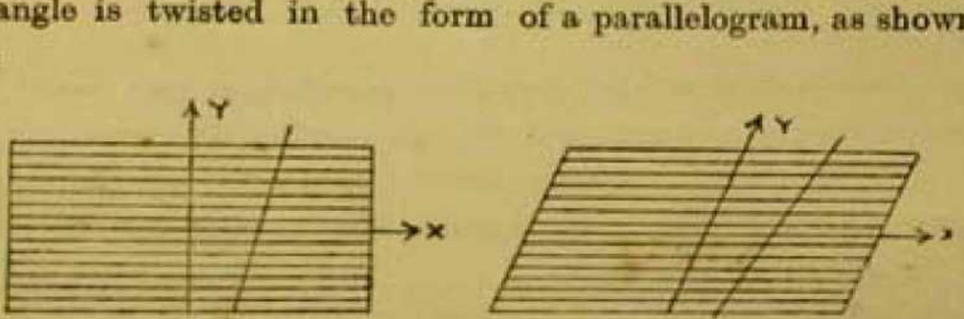
both of the form (7.2). The transformation (7.3) is called a *strain*.

Consider the transformation

$$\begin{aligned}x'&=x+ay\\ y'&=y\end{aligned} \qquad a\neq 0. \tag{7.4}$$

As by (7.4) every point of the x-axis is transformed into itself, so any line intersecting the x-axis is transformed into a line passing through the same point on the axis.

As before, consider a rectangle with its centre at the origin and its sides parallel to the coordinate axes. Draw lines parallel to the x-axis. Twist the rectangle in such a way that every point of the x-axis is left fixed, the parallels are transformed into themselves and any point (x, y), not on the x-axis, is carried along a parallel to the point $(x+ay, y)$, so that the rectangle is twisted in the form of a parallelogram, as shown in the diagram.



If a is positive, any point not on the x-axis slides along a parallel through a distance ay towards the right or the left according as the point is above or below the axis of x. If a is negative, the "right" and the "left" are to be interchanged. If the whole plane is twisted in this manner, we get the above transformation. The transformation (7.4) is called a *simple shear*. It can be verified that area is conserved by a simple shear.

The transformation (7.1) can be factorised in terms of the above particular transformations and translation. Thus if $a_1b_2\neq 0$, the factors may be taken as

$$\left.\begin{aligned}x_1&=x\\ y_1&=\frac{b_1}{b_2}x+y\end{aligned}\right\},\quad \left.\begin{aligned}x_2&=\frac{|a_1b_2|}{b_2}x_1\\ y_2&=b_2y_1\end{aligned}\right\},\quad \left.\begin{aligned}x_3&=x_2+\frac{a_2}{b_2}y_2\\ y_3&=y_2\end{aligned}\right\},\quad \begin{aligned}x'&=x_3+a_0\\ y'&=y_3+b_0,\end{aligned}$$

If $a_1 = b_2 = 0$, the factors may be taken as

$$\left.\begin{array}{l} x_1 = x \\ y_1 = x + y \end{array}\right\},\ \left.\begin{array}{l} x_2 = x_1 - y_1 \\ y_2 = y_1 \end{array}\right\},\ \left.\begin{array}{l} x_3 = x_2 \\ y_3 = x_2 + y_2 \end{array}\right\},\ \left.\begin{array}{l} x_4 = -a_2 x_3 \\ y_4 = b_1 y_3 \end{array}\right\},\ \begin{array}{l} x' = x_4 + a_0 \\ y' = y_4 + b_0 \end{array}$$

23.2 Some invariants. As parallel vectors are transformed by affinity into parallel vectors, a parallelogram is transformed into a parallelogram. Let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be two arbitrary parallelograms. There exists an affinity transforming $\overline{A_1A_2}$ and A_3 to $\overline{B_1B_2}$ and B_3. By this affinity A_4 is transformed into B_4. Hence, every parallelogram is *affine* to every other parallelogram. In particular, every parallelogram is affine to a square.

Let P_1, P_2, P be three collinear points and, by an affinity, let them be transformed into three collinear points P'_1, P'_2, P'.

If $$\overline{P_1P} = \rho\overline{P_2P}, \quad \text{then} \quad \overline{P'_1P'} = \rho\overline{P'_2P'}$$

Hence $$\overline{P_1P}/\overline{P_2P} = \overline{P'_1P'}/\overline{P'_2P'}$$

Thus, *the ratio in which a point divides a line segment remains invariant by affine transformations.* For example, the centroid of a triangle is transformed into the centroid of the transformed triangle. Also, cross-ratio is unaltered by affinity.

Now consider an angle $\pi/2$ between two vectors (a, b) and $(-b, a)$. Let the vectors be transformed by (7.1) into the vectors (a', b') and (a'', b'') respectively. So we have

$$\begin{array}{l} a' = a_1 a + a_2 b \\ b' = b_1 a + b_2 b \end{array} \quad \text{and} \quad \begin{array}{l} a'' = -a_1 b + a_2 a \\ b'' = -b_1 b + b_2 a \end{array}$$

In general, the angle between the transformed vectors will not be $\pi/2$. But if it is $\pi/2$, we must have

$$a'a'' + b'b'' = 0$$

Or $$a^2(a_1a_2 + b_1b_2) + ab(a_2^2 + b_2^2 - a_1^2 - b_1^2) - b^2(a_1a_2 + b_1b_2) = 0. \qquad (7.5)$$

Suppose that the affinity (7.1) is given (that is, the coefficients a_i, b_i are known) while the vector (a, b) is unknown. Then the equation (7.5) may be considered as a quadratic equation in a/b. The discriminant of the equation is

$$(a_2^2 + b_2^2 - a_1^2 - b_1^2)^2 + 4(a_1a_2 + b_1b_2)^2 \geq 0$$

If the equality holds, the affinity is a similarity transforming every pair of orthogonal vectors into a pair of orthogonal vectors. If the inequality holds, we can solve the equation (7.5) and get two real roots; that is, we can obtain two distinct directions. As the product of the roots $= -1$, one direction is obtained from the other by a rotation through $\pm\pi/2$. Thus, *there is one and only one pair of pencils of parallel lines orthogonal to one another which is transformed by a given affinity, which is not similarity, into another pair of pencils of parallel lines orthogonal to one another.*

Further if we suppose that all right angles are transformed into right angles, then (7.5) must be satisfied identically. So

$$a_1a_2+b_1b_2=0$$
$$a_2^2+b_2^2-a_1^2-b_1^2=0$$

From the first of these relations,

$$b_2/a_1=-a_2/b_1=\rho \text{ (say) ; or } \quad b_2=\rho a_1,\ a_2=-\rho b_1$$

Substituting in the second,

$$(\rho^2-1)(a_1^2+b_1^2)=0, \quad \text{Therefore } \rho=\pm 1 \text{ ; hence } \quad b_1=\mp a_2,\ b_2=\pm a_1$$

The affinity now becomes

$$x'=a_1x+a_2y+a_0$$
$$y'=\mp(a_2x-a_1y)+b_0,$$

the upper and the lower signs corresponding to $\rho=+1$ and $\rho=-1$ respectively. These are transformations of similarity, or products of them and orthogonal line reflexions and so all angles are preserved. The above transformations may be factorised into a rigid motion

$$\bar{x}=\frac{1}{|\sqrt{(a_1^2+a_2^2)}|}\,(a_1x+a_2y+a_0)$$

$$\bar{y}=\frac{1}{|\sqrt{(a_1^2+a_2^2)}|}\,(-a_2x+a_1y\pm b_0)$$

and a dilation or a dilation with an orthogonal reflexion

$$x'=|\sqrt{(a_1^2+a_2^2)}|\,\bar{x}$$
$$y'=\pm|\sqrt{(a_1^2+a_2^2)}|\,\bar{y}$$

Thus, if an affinity preserves angles, it can be decomposed into a rigid motion and a dilation with possibly an orthogonal reflexion.

24. Affine classification and properties of nondegenerate conics. Let a nondegenerate conic

$$ax^2+2bxy+cy^2+2dx+2ey+f=0$$

be transformed by the affinity

$$\begin{aligned} x&=p_1x'+p_2y'+p_0 \\ y&=q_1x'+q_2y'+q_0 \end{aligned} \qquad |\,p_1\,q_2\,|\neq 0$$

into $$a'x'^2+2b'x'y'+c'y'^2+2d'x'+2e'y'+f'=0$$

Then

$$a' = ap^2_1 + 2bp_1q_1 + cq^2_1$$

$$b' = ap_1p_2 + b(p_1q_2 + q_1p_2) + cq_1q_2$$

$$c' = ap^2_2 + 2bp_2q_2 + cq^2_2$$

By calculation,

$$\begin{vmatrix} a' & b' \\ b' & c' \end{vmatrix} = \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}^2 \begin{vmatrix} a & b \\ b & c \end{vmatrix}$$

So, a hyperbola, a parabola or an ellipse is transformed into a hyperbola, a parabola or an ellipse respectively. Moreover, a parabola

$$x^2 + 2my = 0$$

can, by the affine transformation $x' = x$, $y' = my$, be transformed into a parabola

$$x'^2 + 2y' = 0$$

And a central conic

$$Ax^2 + By^2 = 1$$

can, by the affine transformation

$$x' = \sqrt{A}\,x, \qquad y' = \sqrt{B}\,y,$$

be transformed into

$$x'^2 + y'^2 = 1, \quad x'^2 - y'^2 = 1 \quad \text{or} \quad x'^2 + y'^2 = -1$$

according as A, B are both positive, A positive and B negative or A, B both negative.

Thus, the equations of the *normal forms* of a parabola, an ellipse, a hyperbola and a second degree curve without real trace are respectively (dropping the dashes)

$$x^2 + 2y = 0, \quad x^2 + y^2 = 1, \qquad x^2 - y^2 = 1, \quad x^2 + y^2 = -1 \qquad (7.6)$$

Hence, it follows from what was stated at the end of § 23 that *all parabolas are affine, all ellipses are affine, and all hyperbolas are affine.* Accordingly there are four nondegenerate conics in affine geometry : the parabola, the ellipse, the hyperbola and the conic without real trace. A circle is regarded as a particular form of an ellipse. Circles, however, are not only affine, but *they are similar*, since any two circles are equivalent with respect to similarity transformations. Also, *all parabolas are similar.*

Let two points A, B of a circle be joined by lines to other points $P_1, P_2, P_3 \ldots\ldots$ of the circle. We then obtain two pencils of lines with their centres at A and B in which there is a one-to-one correspondence between the rays of the pencils such that two corresponding rays intersect on the circle. If the pencils are denoted by (A) and (B), the angles between any two rays AP_r, AP_s of (A) is equal to the angle between the corresponding rays BP_r, BP_s of (B). Therefore the cross-ratio of any four rays of (A) is equal to the cross-ratio of the corresponding four rays of (B). Now, let the circle be transformed by an affinity into an ellipse. Then the pencils (A'), (B') into which (A), (B) are transformed by the same affinity have also the property that the cross-ratio of any four rays of (A') is equal to the cross-ratio of the four corresponding rays of (B'), two rays being corresponding when they intersect on the ellipse. But it does not, however, follow that the angle between two rays of (A') is equal to the angle between the corresponding rays of (B').

This property of equi-cross-ratio is not characteristic of an ellipse alone. As an example, consider the four lines

$$l_1 \equiv \ \ x+y+1=0, \qquad l_2 \equiv x-y+1=0,$$
$$l_3 \equiv -x-y+1=0, \qquad l_4 \equiv x-y-1=0$$

and the two pencils of lines

$$\gamma l_1 + \lambda l_2 = 0, \qquad \gamma l_3 + \lambda l_4 = 0,$$

whose centres are the points of intersections of $l_1=0$, $l_2=0$ and $l_3=0$, $l_4=0$. We put the rays of the two pencils in a one-to-one correspondence by giving the same pair of values to γ, λ in the two pencils. By this correspondence, the cross-ratio of any four rays of one pencil is made equal to the cross-ratio of the four corresponding rays of the other. Now the locus of the points of intersection of the corresponding rays of the two pencils is obtained by eliminating γ, λ between the equations of the two pencils. Thus the locus is

$$\begin{vmatrix} x+y+1 & x-y+1 \\ -x-y+1 & x-y-1 \end{vmatrix} = 0$$

Or

$$x^2-y^2-1=0,$$

which is a hyperbola. It is easily seen that the centres of the pencils also lie on the hyperbola.

24.1 Conjugate diameters. By a *diameter* of a central conic we shall mean either any line passing through the centre of the conic or the segment of this line intercepted by the conic. In the latter case, the length of the segment is called the length of the diameter. By a diameter of a parabola we shall mean any line parallel to the axis of the parabola.

Consider two orthogonal radii OA, OB of the circle $x^2+y^2=1$ and let the coordinates of $\overline{OA}$, $\overline{OB}$ be (x_1, y_1), $(-y_1, x_1)$. Let the circle be transformed into the ellipse

$$x'^2/a^2 + y'^2/b^2 = 1$$

by the transformation

$$x' = ax, \quad y' = by$$

Then the vectors (x_1, y_1), $(-y_1, x_1)$ are transformed into (ax_1, by_1), $(-ay_1, bx_1)$; or, putting $ax_1=\xi$, $by_1=\eta$, $\overline{OA}$, $\overline{OB}$ are transformed into the vectors

$$(\xi, \eta),\ \left(-\frac{a}{b}\eta, \frac{b}{a}\xi\right)$$

respectively. Therefore, the orthogonal lines OA, OB are transformed into the lines

$$\eta x' - \xi y' = 0,\ \ \frac{b}{a}\xi x' + \frac{a}{b}\eta y' = 0$$

That is to say, if the circle $x^2+y^2=1$ be transformed into the ellipse

$$x^2/a^2+y^2/b^2=1,$$

then two arbitrary orthogonal diameters of the circle are transformed into the two diameters

$$\eta x - \xi y = 0, \quad \frac{b}{a}\xi x + \frac{a}{b}\eta y = 0 \tag{7.7}$$

of the ellipse, where ξ, η are arbitrary. Two orthogonal diameters of the circle are called *conjugate diameters of the circle* and it is evident that each of two conjugate diameters of a circle contains the middle points of a system of chords parallel to the other. We can see that this property also holds among the diameters (7.7) of the ellipse, because parallel lines are transformed into parallel lines and middle points of segments into middle points of segments by an affinity. We shall, however, prove this important result directly as follows :

Let $P_0=(x_0, y_0)$ be any point of the ellipse. The coordinates of any point P_1 of the line passing through P_0 and parallel to the vector (ξ, η), *i.e.*, parallel to the first of the lines (7.7) are $(x_0+\rho\xi, y_0+\rho\eta)$. The coordinates of the middle point P of the segment $\overline{P_0P_1}$ are given by

$$x=x_0+\frac{\rho}{2}\xi, \qquad y=y_0+\frac{\rho}{2}\eta$$

If the point P_1 lies on the ellipse, we must have

$$\left(\frac{x_0+\rho\xi}{a}\right)^2+\left(\frac{y_0+\rho\eta}{b}\right)^2=1$$

One value of ρ is zero, since P_0 is on the ellipse. Therefore

$$\rho\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}\right)+2\left(\frac{\xi x_0}{a^2}+\frac{\eta y_0}{b^2}\right)=0$$

or

$$\rho/2=-(b^2\xi x_0+a^2\eta y_0)/(b^2\xi^2+a^2\eta^2)$$

Substituting the value of $\rho/2$ in the coordinates of P, we have

$$x = a^2\eta(\eta x_0 - \xi y_0)/(b^2\xi^2 + a^2\eta^2)$$

$$y = -b^2\xi(\eta x_0 - \xi y_0)/(b^2\xi^2 + a^2\eta^2)$$

Therefore the locus of P, obtained by eliminating x_0, y_0, is

$$b^2\xi x + a^2\eta y = 0, \quad \text{or} \quad \frac{b}{a}\xi x + \frac{a}{b}\eta y = 0$$

Hence, the second of the two diameters of (7.7) contains the middle points of chords parallel to the first. It can be seen similarly that this property remains true if we interchange the diameters. Two diameters of an ellipse, each of which contains the middle points of the system of chords parallel to the other, are called *conjugate diameters of the ellipse.* Thus, two conjugate diameters of a circle are tronsformed into two conjugate diameters of the ellipse into which the circle is transformed by an affinity.

The first of the conjugate diameters (7.7) meets the ellipse in the points

$$\left(\frac{ab\xi}{\pm\sqrt{C}}, \frac{ab\eta}{\pm\sqrt{C}}\right), \qquad \text{where} \quad C = a^2\eta^2 + b^2\xi^2$$

It may easily be verified that the tangents to the ellipse at these points are parallel to the second of the conjugate diameters. Thus, *the tangent at any extremity of one of two conjugate diameters of an ellipse is parallel to the other.*

As an *application*, suppose, as before, that the given circle is transformed into the ellipse by the given affinity. Any square inscribed in the circle is transformed into a parallelogram inscribed in the ellipse. As the diagonals of the square are conjugate diameters of the circle, so the diagonals of the parallelogram are also conjugate diameters of the ellipse. Also, the area of the parallelogram is ab times the area of the square by virtue of the given transformation. Since all squares inscribed in the given circle have the same area and ab is constant, therefore all parallelograms inscribed in a given ellipse whose diagonals are conjugate diameters have the same area.

The slopes (§4) λ, λ' of the two conjugate diameters given by (7.7) are

$$\eta/\xi \text{ and } \quad -b^2\xi/a^2\eta$$

Therefore, for any pair of conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we have

$$\lambda\lambda' = -b^2/a^2 \tag{7.7'}$$

The axes of an ellipse are the only pair of conjugate diameters which are orthogonal, unless the ellipse is a circle. For, if there exists a pair of orthogonal conjugate diameters, other than the axes, we must have $\lambda\lambda' = -1$; therefore $b^2 = a^2$. Further, if there exists a pair of conjugate diameters of the ellipse which are equally inclined to the axes, we must have $\lambda = -\lambda'$. So,

$$\lambda^2 = b^2/a^2, \quad \text{or} \quad \lambda = \pm b/a$$

The equations of this pair of conjugate diameters are

$$bx - ay = 0, \qquad bx + ay = 0$$

Therefore, they are the diagonals of the rectangle circumscribed about the ellipse.

In the case of a hyperbola $x^2/a^2 - y^2/b^2 = 1$, take a diameter $\eta x - \xi y = 0$. Then, as in the case of an ellipse, it may be seen that each of the two diameters

$$\eta x - \xi y = 0, \quad \frac{b}{a}\xi x - \frac{a}{b}\eta y = 0 \tag{7.8}$$

of the hyperbola contains the middle points of the system of chords parallel to the other. Two diameters of a hyperbola possessing this property are called *conjugate diameters of the hyperbola*. As in the case of an ellipse, the *tangent at any extremity of one of two conjugate diameters of a hyperbola is parallel to the other*.

The slopes λ, λ' of the two conjugate diameters (7.8), other than the axes, are

$$\eta/\xi \quad \text{and} \quad b^2\xi/a^2\eta$$

Therefore, for any pair of conjugate diameters of the hyperbola,

$$\lambda\lambda' = b^2/a^2 \tag{7.8'}$$

It follows that two conjugate diameters are equally inclined to each asymptote and are separated by it. If there exists a pair of conjugate diameters each of which has the same inclination to one axis as the other has to the other axis, we must have $a = b$, so that the hyperbola is rectangular. The asymptotes are then $x + y = 0$, $x - y = 0$ and a pair of conjugate diameters are

$$\eta x - \xi y = 0, \quad \xi x - \eta y = 0.$$

Thus, *the asymptotes bisect the angles between any pair of conjugate diameters of a rectangular hyperbola.*

If two conjugate diameters of a hyperbola coincide, it follows from (7.8) that

$$\begin{vmatrix} \eta & \xi \\ \frac{b}{a}\xi & \frac{a}{b}\eta \end{vmatrix} = 0,$$

or $$\xi^2/\eta^2 = a^2/b^2, \qquad \text{Therefore } \xi/\eta = \pm a/b.$$

That is to say, a diameter which is its own conjugate is either $bx - ay = 0$, or $bx + ay = 0$. But these are the asymptotes of the hyperbola. Thus, *each of the two asymptotes is a self-conjugate diameter of a hyperbola*. Strictly speaking, however, an asymptote has no conjugate, because there is no chord parallel to it.

In the case of a parabola, $y^2 = 2px$, take a vector (ξ, η) not parallel to the axis of the parabola. Then, as in the case of an ellipse or a hyperbola, it may be seen that the middle points of a system of chords parallel to (ξ, η) lie on the line

$$y = p\frac{\xi}{\eta} \tag{7.9}$$

But p and ξ/η are constants for the given system of parallel chords of the given parabola. Therefore, the middle points of a system of parallel chords of a parabola lie on a line parallel to the axis of the parabola, that is, *on a diameter of the parabola*. As in the case of an ellipse and a hyperbola, *the tangent at the extremity of a diameter of a parabola is parallel to the system of chords bisected by the diameter.*

Consider four diameters of an ellipse (or of a hyperbola)

$$\eta_1 x - \xi_1 y = 0, \qquad \eta_2 x - \xi_2 y = 0,$$
$$(\gamma\eta_1 + \lambda\eta_2)x - (\gamma\xi_1 + \lambda\xi_2)y = 0, \qquad (\gamma'\eta_1 + \lambda'\eta_2)x - (\gamma'\xi_1 + \lambda'\xi_2)y = 0.$$

The diameters which are conjugate respectively to these lines are, by (7.7),

$$\frac{b}{a}\xi_1 x + \frac{a}{b}\eta_1 y = 0, \qquad \frac{b}{a}\xi_2 x + \frac{a}{b}\eta_2 y = 0,$$

$$(\gamma\xi_1 + \lambda\xi_2)\frac{b}{a}x + (\gamma\eta_1 + \lambda\eta_2)\frac{a}{b}y = 0, \qquad (\gamma'\xi_1 + \lambda'\xi_2)\frac{b}{a}x + (\gamma'\eta_1 + \lambda'\eta_2)\frac{a}{b}y = 0$$

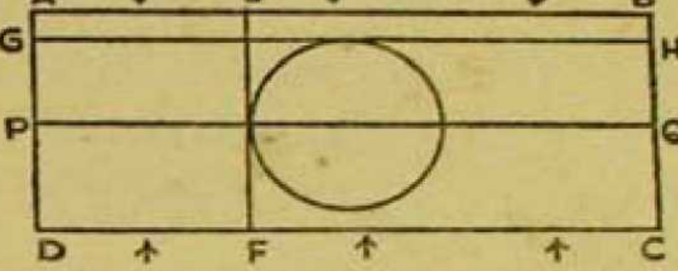
Denoting the first four lines by g_i and the last four lines by g'_i, $i = 1, 2, 3, 4$, we have the cross-ratio

$$(g_1\, g_2,\, g_3\, g_4) = (g'_1\, g'_2,\, g'_3\, g'_4) = \gamma'\lambda/\lambda'\gamma$$

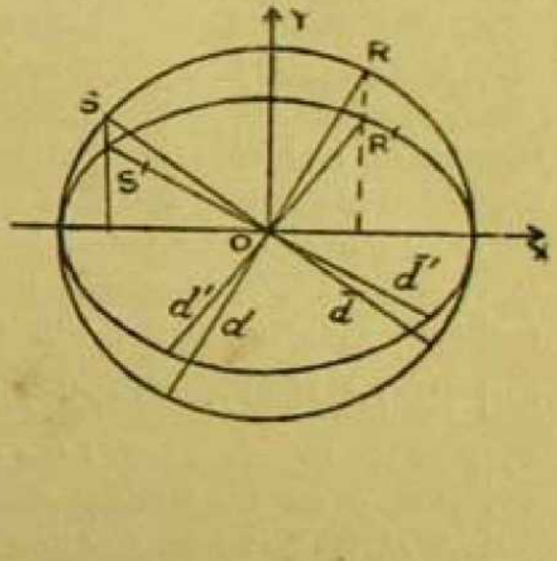
Thus, *the cross-ratio of any four diameters of a central conic is equal to the cross-ratio of the four conjugate diameters.*

24.2 Transformation of conjugate diameters by simple strain.

(1) *Ellipse.* Consider a flat rectangular bar $ABCD$ of any elastic uniform material on one face of which lines and a circle are drawn, as in the figure. We imagine that the bar is subjected to pressures on the two ends AB, DC orthogonally while their lengths are kept unchanged, and that there is no bulging of the faces due to pressure. All segments EF parallel to AD will then remain fixed in position but their lengths will be shortened to $E'F'$ in the same ratio $r = |E'F'/EF|$, say. All segments GH parallel to AB will be carried into parallel segments, with the exception of one segment PQ which will remain fixed, and their lengths will remain constant.



Let the line PQ be taken as the axis of x. A line p drawn on the face with a slope $\tan\phi$ will be carried into a line p' with a slope $r\tan\phi$. The transformation of the kind considered here is a simple strain or compression (§ 23.1). For the sake of simplicity, let the equation of the circle be $x^2+y^2=a^2$ and let d be a diameter of the circle with slope $\tan\phi$.



Then the conjugate diameter $\bar{d}$ has the slope $\tan(\pi/2+\phi)$ or $-\cot\phi$. Since a point $R=(x, y)$ of the circle is carried into the point $R'=(x, ry)$, the equation of the circle is transformed into

$$x^2/a^2+y^2/b^2=1, \text{ where } b=ra.$$

Thus, the circle is transformed into an ellipse ; and d', $\bar{d}'$ into which d, $\bar{d}$ are carried have slopes $r\tan\phi$, $-r\cot\phi$ respectively. Therefore, the product of the slopes of d', $\bar{d}'$ is equal to

$$-r^2=-b^2/a^2$$

Hence, by (7.7′), d', $\bar{d}'$ are conjugate diameters of the ellipse, as is to be expected.

The circle described on the major axis of an ellipse as a diameter is called *the auxiliary circle of the ellipse.* The two points R, R' of the circle and the ellipse which are respectively the extremities of d and d', having the same x coordinate, are called *corresponding points*, and the angle ϕ is called the *eccentric angle* for the point R' of the ellipse. The coordinates of R' are therefore

$$(a \cos \phi, \quad b \sin \phi)$$

in terms of the parameter ϕ. Thus the parametric equations of an ellipse are the same as given by (4.15). The coordinates of the extremities R' and S' of two conjugate diameters of the ellipse are

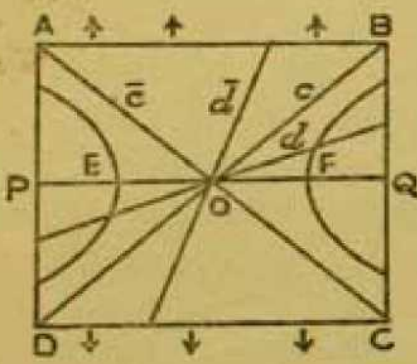
$$(a \cos \phi,\ b \sin \phi) \quad \text{and} \quad (-a \sin \phi,\ b \cos \phi)$$

Therefore, the sum of the squares of half the lengths of any two conjugate diameters is equal to

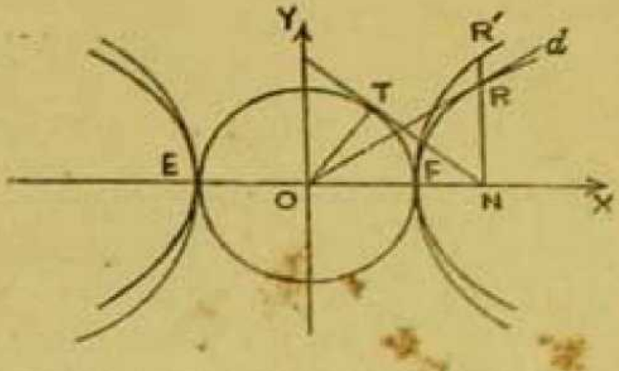
$$|O R'|^2 + |O S'|^2 = a^2 + b^2$$

and is therefore equal to the sum of the squares of half the lengths of the axes.

(2) *Hyperbola* : Consider a square lamina $ABCD$ of any uniform elastic material. On the lamina draw the diagonals c, c, two lines d, d through the centre O equally inclined to the diagonals, the line PQ through the centre O parallel to AB and the portion of the rectangular hyperbola of which the lines c, c are the asymptotes. We imagine that the lamina is subjected to tensions at the two ends AB, DC while the lengths of segments parallel to AB are kept constant, the line PQ is kept fixed in position and that there is no bulging. The lamina will then be elongated into a rectangle ; lines parallel to AB, other than PQ, will move parallel to themselves ; lines parallel to AD will remain fixed in position but their lengths will be increased to $A'D'$ in the same ratio $r = |A'D'/AD|$, say. The diagonals $c, \bar{c}$ will be carried into the diagonals $c', \bar{c}'$ of the rectangle. Let the line PQ be taken as the axis of x. Then any line p on the lamina with a slope λ will be carried into a line p' with a slope $r\lambda$. The transformation of the kind considered here is a simple elongation (§ 23.1).



The rectangular hyperbola with its asymptotes $c, \bar{c}$ and its conjugate diameters $d, \bar{d}$ will be carried into a hyperbola with its asymptotes $c', \bar{c}'$ and its conjugate diameters $d', \bar{d}'$. For the sake of simplicity, let the equation of the rectangular hyperbola be $x^2 - y^2 = a^2$. Since a point $R = (x, y)$ of the rectangular hyperbola is carried into a point $R' = (x, ry)$, the equation of the rectangular hyperbola is transformed into $x^2/a^2-y^2/b^2=1$, where $b=ra$.



If the slope of d is $\tan\phi$, then that of d is $\tan(\pi/2-\phi)$ or $\cot\phi$; therefore, the slopes of d', d' are $r\tan\phi$ and $r\cot\phi$ and their product is equal to

$$r^2=b^2/a^2.$$

Hence, by (7.8′), d', d' are conjugate diameters of the transformed hyperbola, as is to be expected. Similarly, the asymptotes are transformed into asymptotes.

As in the figure, the circle described on the axis EF of a hyperbola as a diameter is the *auxiliary circle of the hyperbola*. If R is a point of the hyperbola, N the foot of the perpendicular from R to EF and NT the tangent to the auxiliary circle touching it at T, then the angle ϕ between the positive x-axis and OT is called the *eccentric angle* for the point R. The coordinates of R', the point corresponding to R, are then

$$(a\sec\phi,\quad b\tan\phi)$$

in terms of the parameter ϕ as given in (4.16). The two hyperbolas have the same auxiliary circle and we have the same eccentric angle for the corresponding points R and R'.

CHAPTER VIII

INVOLUTION

25. Projective pencils of lines. Let the equations of two concurrent lines g_1 and g_2 be $l_1(x, y)=0$ and $l_2(x, y)=0$ respectively. It has been shown in § 7 that these two lines determine a pencil, and that if g, g' be any two lines of the pencil, their equations can be written as $\gamma l_1+\lambda l_2=0$, $\gamma' l_1+\lambda' l_2=0$ and that the cross-ratio of the four has the following value :

$$(g_1g_2, gg')=\lambda\gamma'/\gamma\lambda'.$$

We give below the formula for the cross-ratio of *any four* lines of the pencil. Let the equation of a line g_k be $\mu_k l_1+\nu_k l_2=0$; similarly for three other lines g_m, g_i, g_j. Suppose we represent the equations of g_k, g_m by new symbols $f_1=0, f_2=0$ respectively, *i.e.*,

$$\mu_k l_1+\nu_k l_2=f_1, \qquad \mu_m l_1+\nu_m l_2=f_2$$

If we solve these equations for l_1 and l_2 in terms of f_1 and f_2, then

$$\rho l_1=\nu_m f_1-\nu_k f_2, \qquad \rho l_2=\mu_k f_2-\mu_m f_1, \quad \text{where} \quad \rho=\mu_k\nu_m-\nu_k\mu_m$$

Substituting these values of l_1 and l_2 in the equations of g_i, g_j and remembering that the equations involved are linear which, when equated to zero, represent lines, the equations of g_k, g_m, g_i, g_j can now be written respectively as

$$f_1=0, \quad f_2=0,$$

$$(\mu_i\nu_m-\nu_i\mu_m)f_1-(\mu_i\nu_k-\nu_i\mu_k)f_2=0, \qquad (\mu_j\nu_m-\nu_j\mu_m)f_1-(\mu_j\nu_k-\nu_j\mu_k)f_2=0$$

Hence applying the above result, we have

$$(g_kg_m, g_ig_j)=\frac{(\mu_i\nu_k-\nu_i\mu_k)\,(\mu_j\nu_m-\nu_j\mu_m)}{(\mu_i\nu_m-\nu_i\mu_m)\,(\mu_j\nu_k-\nu_j\mu_k)} \tag{8.1}$$

Let there be another pencil of lines and let the lines of this pencil be denoted by g''s. If we choose any three lines g_i, g_j, g_k of the first pencil to correspond respectively to any three lines g'_i, g'_j, g'_k of the second, then we can set up a correspondence between the lines of the two pencils in such a manner that g_m and g'_m are corresponding lines if and only if

$$(g_ig_j, g_kg_m)=(g'_ig'_j, g'_kg'_m).$$

This correspondence is called a *projectivity* and the two pencils are said to be *projective*. In a projectivity, to any line of one pencil there corresponds a definite line of the other pencil, and the cross-ratio of any four lines of one pencil is equal to the cross-ratio of the corresponding four lines of the other pencil, this cross-ratio being determined by (8.1).

In two projective pencils, let the three different lines g_1, g_2, g_3 with equations $l_1=0$, $l_2=0$, $l_1+l_2=0$ of one pencil correspond respectively to the three lines g'_1, g'_2, g'_3 with equations $l'_1=0$, $l'_2=0$, $l'_1+l'_2=0$ of the other pencil. As l_1 and l_2 may take arbitrary factors, we can always so arrange that the lines g_1, g_2, g_3 are represented in this manner. Then the lines g and g' of the two pencils whose equations are $\mu l_1+\nu l_2=0$, $\mu l'_1+\nu l'_2=0$ are corresponding lines, because

$$(g_1g_2, g_3g)=\mu/\nu=(g'_1g'_2, g'_3g').$$

If any two corresponding lines have a point in common, that point must satisfy the equation

$$\begin{vmatrix} l_1 & l_2 \\ l'_1 & l'_2 \end{vmatrix}=0$$

Since the l's and l''s are linear functions, this equation is generally of the second degree in x, y and so represents a conic. We shall however take up this question of projective generation of conics in a subsequent chapter.

Two pencils of lines which have the same centre are called *concentric* pencils. If two nonconcentric projective pencils are such that their corresponding lines intersect on a line, then the pencils are said to be *perspective*. As has been shown in § 7.1, perspectivity is a special case of projectivity, because cross-ratio is unaltered by projection. Two nonconcentric projective pencils are perspective if and only if the line joining their centres corresponds to itself.

Consider two concentric pencils. Each line through the common centre can now be regarded as belonging to either pencil. Let there be a projectivity between the pencils in which the lines g_1, g_2, g_3, g' considered as belonging to one pencil correspond respectively to the lines g_k, g_m, g_l, g considered as belonging to the other. So,

$$(g_1g_2, g_3g')=(g_kg_m, g_lg)$$

Or, with the same equations as given before, we have by (8.1)

$$\frac{\mu'}{\nu'}=\frac{(\mu_l\nu_k-\nu_l\mu_k)\,(\mu\nu_m-\nu\mu_m)}{(\mu_l\nu_m-\nu_l\mu_m)\,(\mu\nu_k-\nu\mu_k)}$$

Suppose that the projectivity has been established by the three pairs of lines (g_1, g_k), (g_2, g_m), (g_3, g_l) ; that is, suppose that the constants in the equations of g_k, g_m, g_l are known. The above equation can then be written as

$$\begin{aligned} \rho\mu' &= a_1\mu + a_2\nu \\ \rho\nu' &= b_1\mu + b_2\nu, \end{aligned} \tag{8.2}$$

where ρ is an arbitrary constant, other than zero, and the a's, the b's are known constants. The equations (8.2) constitute a linear transformation, $g \rightarrow g'$, between the lines of the two pencils. As this is a one-to-one correspondence, we must have

$$a_1b_2 - b_1a_2 \neq 0.$$

So, the inverse, $g' \rightarrow g$, is given by

$$\begin{aligned} \sigma\mu &= -b_2\mu' + a_2\nu' \\ \sigma\nu &= b_1\mu' - a_1\nu' \end{aligned}$$

Eliminating ρ between the equations (8.2),

$$b_1\mu\mu' + b_2\nu\mu' - a_1\mu\nu' - a_2\nu\nu' = 0 \tag{8.3}$$

Equations of the form (8.2) or (8.3), where the a's and the b's are constants, are known as the bilinear or *homographic* transformations.

26. Involution of lines. In a projectivity between two concentric pencils, a line g considered as belonging to one pencil corresponds to a line g' belonging to the other. But we can also consider g and g' as belonging to the second and the first pencils respectively. When this is so, it follows from (8.2) and its inverse that, in general, g and g' are not corresponding lines in the projectivity. In the special case when g and g' correspond to one another doubly, *i.e.*, $g \longleftrightarrow g'$, the projectivity is known as an *involution*.

Suppose that the projectivity considered above is an involution. Then, interchanging μ, μ' and ν, ν', we have, from (8.3), the two equations

$$b_1\mu\mu' + b_2\nu\mu' - a_1\mu\nu' - a_2\nu\nu' = 0$$

$$b_1\mu'\mu + b_2\nu'\mu - a_1\mu'\nu - a_2\nu'\nu = 0$$

Subtracting,

$$(b_2 + a_1)(\nu\mu' - \nu'\mu) = 0$$

As $\nu\mu' - \nu'\mu \neq 0$, because the lines g, g' are distinct,

$$b_2 + a_1 = 0.$$

This condition is independent of (μ,ν) and (μ',ν'). Therefore, if to any line g_r considered as a line of the first pencil corresponds a line g_s of the second, then to g_s considered as belonging to the first pencil corresponds g_r considered as belonging to the second; *i.e.*, $g_r \longleftrightarrow g_s$. Therefore, also $g_1 \longleftrightarrow g_k$, $g_2 \longleftrightarrow g_m$, $g_3 \longleftrightarrow g_l$. Accordingly, in an involution *lines of a pencil are* *paired off in a definite way.* Two corresponding lines of an involution are also called *conjugate* lines.

Since the condition $b_2 + a_1 = 0$ holds, the equation of an involution can be written, by (8.3), as

$$c_1\mu\mu' + c_2(\mu\nu' + \nu\mu') + c_3\nu\nu' = 0 \quad \text{where} \quad c_1c_3 - c_2^2 \neq 0 \tag{8.4}$$

From (8.4) it follows that two such equations are necessary to determine the ratio $c_1 : c_2 : c_3$. That is to say, an involution is determined by two pairs of corresponding lines. This can also be seen as follows :

Suppose (g_a, g'_a) and (g_b, g'_b) are to be two pairs of corresponding lines of an involution. Since a projectivity can be established by three pairs of corresponding lines, we determine the projectivity in which the three lines g_a, g_b, g'_a correspond to the three lines g'_a, g'_b, g_a respectively. This projectivity is the required involution, because $g_a \longleftrightarrow g'_a$

Let us now enquire whether there is any line which corresponds to itself in the involution. If the two corresponding lines g, g' coincide, we can write

$$\mu = \rho\mu', \quad \nu = \rho\nu'$$

So, from (8.4),

$$c_1\mu^2 + 2c_2\mu\nu + c_3\nu^2 = 0 \tag{8.5}$$

The discriminant of this equation is $-4\,(c_1c_3 - c_2^2)$. There are two possibilities :

(1) If $c_1c_3 - c_2^2 < 0$, the discriminant is positive ; and so there are two self-corresponding lines which are called the *double lines* of the involution and the involution is called a *hyperbolic* involution.

(2) If $c_1c_3 - c_2^2 > 0$, the discriminant is negative ; and so there is no (real) double line and the involution is called an *elliptic* involution.

In an involution let (g_1, g_2) and (g, g') be two pairs of corresponding lines whose equations are, as before,

$$l_1 = 0,\ l_2 = 0 \quad \text{and} \quad \mu l_1 + \nu l_2 = 0,\ \mu' l_1 + \nu' l_2 = 0$$

Since (8.4) must be satisfied by the constants (1,0), (0,1) in the equations of g_1, g_2, we must have $c_2 = 0$. Thus the *normal form* of the equation of an involution is

$$c_1\mu\mu' + c_3\nu\nu' = 0, \quad c_1c_3 \neq 0 \tag{8.6}$$

and the involution is hyperbolic or elliptic according as $c_1c_3 \lesseqgtr 0$. From (8.6) we have

$$\mu\mu'/\nu\nu' = -c_3/c_1 = -c_1c_3/c_1^2 \ ;$$

therefore the involution is hyperbolic or elliptic according as $\mu\mu'/\nu\nu' \gtreqless 0$; but $\mu\mu'/\nu\nu'$ has the same sign as $(\mu\mu'/\nu\nu')(\nu^2/\mu^2) = \nu\mu'/\mu\nu' = (g_1\, g_2,\, g\, g')$. So, the involution is hyperbolic or elliptic according as the cross-ratio $(g_1\, g_2,\, g\, g') \gtreqless 0$, where (g_1, g_2) and (g, g') are any two pairs of corresponding lines. Thus it follows that *in an elliptic involution any pair of corresponding lines are separated by any other pair of corresponding lines, but in a hyperbolic involution pairs of corresponding lines are not so separated.*

In a hyperbolic involution, let the double lines be chosen as the fundamental lines g_1, g_2 with equations $l_1=0$, $l_2=0$ and let the lines g, g' with equations

$$\mu l_1 + \nu l_2 = 0, \quad \mu' l_1 + \nu\, l_2 = 0$$

be a pair of corresponding lines. Since the constants in the equation of each of the double lines must statisfy (8.5), we have $c_1 = c_3 = 0$. Therefore, the normal form of the equation of a hyperbolic involution is, by (8.4),

$$\mu\nu' + \mu'\nu = 0, \tag{8.7}$$

$$\therefore \qquad \nu\mu'/\mu\nu' = -1, \quad \text{or} \quad (g_1\, g_2,\, gg') = -1$$

Thus, *the double lines of a hyperbolic involution are harmonically separated by every pair of corresponding lines.* If, in particular, the double lines are orthogonal to one another, then any two corresponding lines are equally inclined to the double lines.

Now, suppose that two involutions have a pair of corresponding lines in common. If the equation of one of these involutions be in the general form (8.4), the equation of the other can be thrown into the normal form (8.6). Let the equations of the involutions be

$$a_1\mu\mu' + a_2(\mu\nu' + \nu\mu') + a_3\nu\nu' = 0$$

and

$$b_1\mu\mu' + b_2\nu\nu' = 0$$

Put $\mu/\nu = \xi$. Then, from the second equation,

$$\mu'/\nu' = -b_2/b_1\xi$$

Substituting this value of μ'/ν' in the first equation,

$$a_1\xi + a_2\left\{-\frac{b_1}{b_2}\xi^2 + 1\right\} + a_3\left(-\frac{b_1}{b_2}\xi\right) = 0$$

or

$$-\frac{b_1}{b_2}a_2\xi^2 + \left(a_1 - \frac{b_1}{b_2}a_3\right)\xi + a_2 = 0.$$

The discriminant of this quadratic equation is

$$\left(a_1 - \frac{b_1}{b_2}a_3\right)^2 + 4\,\frac{b_1}{b_2}a_2^{\,2};$$

this is positive if $b_1/b_2 > 0$; and $b_1/b_2 > 0$ implies that the second involution is elliptic. Thus, *two concentric involutions of lines, of which at least one is elliptic, have a common pair of corresponding lines.*

Every hyperbolic involution contains just one pair of corresponding lines which are orthogonal, namely, the bisectors of the angles between the double lines. If at least two pairs of corresponding lines of an involution are orthogonal, then any other pair of corresponding lines are also orthogonal. For, let the equations of two pairs of corresponding lines be

$$l_1 = 0,\ l_2 = 0 \quad \text{and} \quad \mu l_1 + \nu l_2 = 0,\ \mu' l_1 + \nu' l_2 = 0$$

Then the equation of the involution is given by (8.6), namely,

$$c_1 \mu\mu' + c_2 \nu\nu' = 0$$

But then if the two lines $l_1 = 0$, $l_2 = 0$ as well as the two lines $\mu l_1 + \nu l_2 = 0$, $\mu' l_1 + \nu' l_2 = 0$ are orthogonal, we must have $c_1 = c_2$. Therefore the equation of the involution becomes

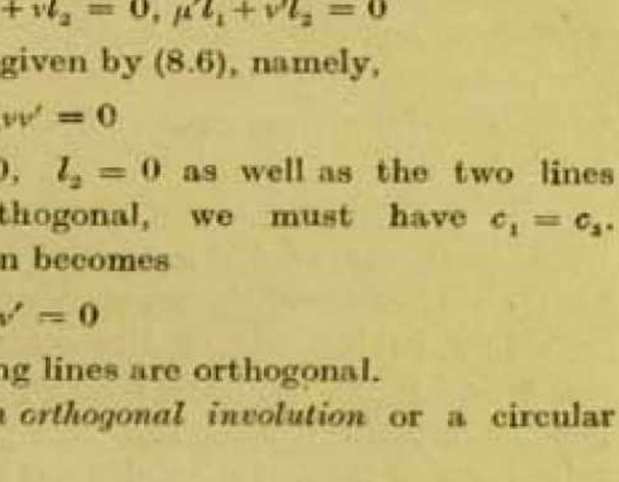
$$\mu\mu' + \nu\nu' = 0$$

which shows that any two corresponding lines are orthogonal.

This kind of involution is called *an orthogonal involution* or a circular involution, and evidently it is elliptic.

If two pencils with different centres are projective or, in particular, perspective, an involution in one pencil gives rise to a corresponding involution in the other. Every involution perspective to a hyperbolic (elliptic) involution is hyperbolic (elliptic).

Every elliptic involution can be put in perspective to an orthogonal involution. For, let two pairs of corresponding lines of a given elliptic involution cut a transversal in the points (A, A') (B, B'). Describe two circles on AA' and BB' as diameters. If P be one of the points of intersection of the two circles, the lines (PA, PA') and (PB, PB') are two pairs of corresponding lines of an orthogonal involution perspective to the given elliptic involution. Every hyperbolic involution is perspective to a symmetry.



27. Projective rows. Involution of points. As in the last article, we can determine the cross-ratio of any four points of a row of points and can establish a *projectivity* between two rows by choosing any three points G_a, G_b

G_c of one row to correspond to any three points G'_a, G'_b, G'_c of the other. Two points G, G' of the two rows correspond to one another in this projectivity if

$$(G_a\ G_b\ G_c\ G) = (G'_a\ G'_b,\ G'_c\ G')$$

There is however one exceptional case as has been noticed in § 7.1. In a given projectivity, there are, in general, two particular points, one of each row, for which the corresponding points do not exist.

Two rows which have the same base are called *cabasal* rows. If two projective rows with different bases are such that the lines joining the corresponding points are concurrent, the rows are said to be *perspective*. Perspective rows are special projective rows. Two projective rows whose bases intersect are perspective if and only if the point of intersection corresponds to itself.

If the projectivity between two cobasal rows are such that a point A, other than one of the particular points in the exceptional case mentioned above, has the same corresponding point A', whether A is regarded as belonging to the one or the other row, *i.e.*, if two corresponding points correspond to one another doubly, $A \leftarrow \rightarrow A'$, then the points of every pair correspond doubly and the projectivity is called an *involution* of points on a line. In an involution of points, the points are paired off in a definite way. There is, however, the exception that in an involution there is, a particular point whose corresponding point does not exist. This particular point is called *the centre of the involution*. Consider an involution of lines in which (a, a'), (b, b'),......are pairs of corresponding lines and let a transversal p meet a, a', b, b',......in A, A', B, B',...... Then (A, A'), (B, B'),......are pairs of corresponding points of an involution of points. There is, in the involution of lines, a line g' parallel to p; if the line g of the pencil corresponding to g' meets p in G, then G is the centre of the involution of points.

An involution of points is determined by two pairs of corresponding points. Also, as in the last article, an involution of points is of two kinds: hyperbolic and elliptic. Hyperbolic involutions have two double points which separate every pair of corresponding points harmonically. Elliptic involutions have no (real) double points and any pair of corresponding points are separated by any other.

Let A, A', B, B', C, C' be six fixed points of a line and X, X' two other points of the same line, Let the coordinates of these points from a chosen origin on the line (§ 1) be $a, a', b, b', c, c', x, x'$ respectively. If (X, X') be a pair of corresponding points of the projectivity determined by the three pairs (A, A'), (B, B'), (C, C'), then

$$(AB, CX) = (A'B', C'X')$$

or
$$\frac{(a-c)\,(b-x)}{(b-c)\,(a-x)} = \frac{(a'-c')\,(b'-x')}{(b'-c')\,(a'-x')}$$

As the quantities a, a', b, b', c, c' are constants, the above equation can be written as

$$pxx' + qx + rx' + s = 0, \tag{8.8}$$

where, p, q, r, s are constants. This is a projective or *homographic* transformation of points on a line provided that the left-hand side is not resoluble into factors, the condition for which is $ps - qr \neq 0$.

If the projectivity is an involution, then, as in the last article, $q = r$. So, the equation of an involution of points is

$$pxx' + q(x + x') + s = 0 \tag{8.9}$$

The double points of a hyperbolic involution are given by the roots of the equation

$$px^2 + 2qx + s = 0, \tag{8.9$'$}$$

provided that the roots are real. Equation (8.9) can be written as

$$\left(x + \frac{q}{p}\right)\left(x' + \frac{q}{p}\right) = \frac{q^2}{p^2} - \frac{s}{p}, \quad p \neq 0$$

If the coordinate of one of the points X, X' is $-q/p$, the other point cannot be determined. Therefore if O be the point whose coordinate is $-q/p$, then O is the centre of the involution. Changing the origin to O, the equation takes the form

$$xx' = k, \tag{8.10}$$

where k is a constant. Thus, *the centre O of an involution has the property*

$$\overline{OA}.\,\overline{OA'} = \overline{OB}.\,\overline{OB'} = \overline{OC}.\,\overline{OC'} = \ldots\ldots, \tag{8.11}$$

where (A, A'), (B, B'), (C, C'), are pairs of corresponding points of the involution. The converse is also true, namely that if (8.11) holds then (A, A'), (B, B'),......are pairs of corresponding points of an involution. This may be proved by retracing the steps.

If k is a positive constant, there are two double points given by

$$x^2 = k, \quad \text{or} \quad x = \pm\sqrt{k},$$

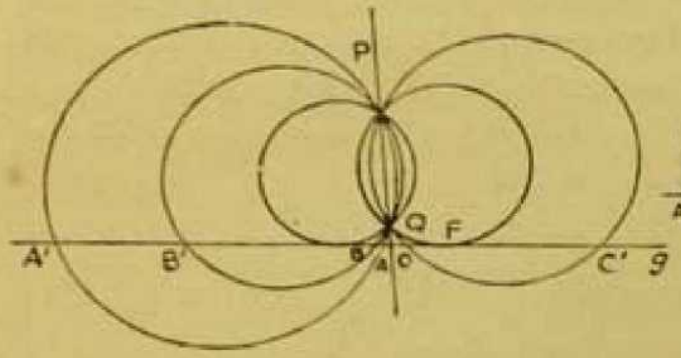
and the involution is hyperbolic. Let E, F be the double points. Since E, F separate any two corresponding points harmonically, the centre of the involution is the middle point of the segment EF and

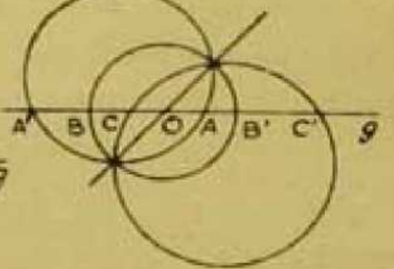
$$k = |\,OE\,|^2 = |\,OF\,|^2.$$

If k is a negative constant, there is no (real) double point and the involution is elliptic. The centre of the involution lies between any two corresponding points.

Applications. (1) To find by construction an involution when two pairs of corresponding points are given.

Let (A, A'), (B, B') be two pairs of given points on a line g and P a point outside g. Draw two circles through A, A', P and B, B', P and





let the circles intersect in a second point Q. Let the line PQ meet g in O. If an arbitrary circle of the pencil of circles drawn through the points P, Q meet g in C, C', then

$$\overline{OP}.\overline{OQ} = \overline{OA}.\overline{OA'} = \overline{OB}.\overline{OB'} = \overline{OC}.\overline{OC'}$$

Hence, by (8.11), (A, A'), (B, B'), (C, C') are pairs of corresponding points of an involution of which O is the centre. If O lies outside the segment PQ, then there are two circles in the pencil which touch g in the points E, F. The involution is then hyperbolic and E, F are the double points. Otherwise the involution is elliptic.

(2) Since the power of a point on the radical axis is the same for all circles of a coaxal system, *any transversal is cut by a system of coaxal circles in pairs of points of an involution.*

27.1 Involution of conjugate points and lines with respect to a nondegenerate conic. The polars of collinear points with respect to a conic are evidently concurrent lines, and conversely the poles of concurrent lines are collinear points. Moreover, the cross-ratio of four collinear points is equal to the corresponding cross-ratio of their polars with respect to a conic. For let

$$A_1 = (x_1, y_1),\ A_2 = (x_2, y_2),\ A = (\gamma x_1 + \lambda x_2,\ \gamma y_1 + \lambda y_2),\ \gamma + \lambda = 1,$$

be any three collinear points. If the equations of the polars of A_1 and A_2 with respect to a conic be $l_1(x, y) = 0$ and $l_2(x, y) = 0$, then the equation of the polar of A with respect to the conic is seen to be $\gamma l_1 + \lambda l_2 = 0$, where the l's are linear functions. This shows that the polars are concurrent. Moreover, if

$$A' = (\gamma' x_1 + \lambda' x_2, \gamma' y_1 + \lambda' y_2), \quad \gamma' + \lambda' = 1,$$

is any other collinear point, the equation of the polar of A' is $\gamma' l_1 + \lambda' l_2 = 0$. Therefore if a_1, a_2, a, a' are the polars of A_1, A_2, A, A' with respect to the conic, then the cross-ratio

$$(A_1A_2, AA') = (a_1a_2, aa') = \lambda\gamma'/\lambda'\gamma$$

Clearly, if the collinear points lie on a line g, their polars pass through the pole of g. Now, let (A, A'), (B, B') be pairs of conjugate points of a line g with respect to a conic and let G be the pole of g. If the lines a, a', b, b' are the polars of A, A', B, B' respectively, then a, a', b, b' are the lines GA', GA, GB', GB We then have

$$(AA', BB') = (aa', bb') \quad \text{and} \quad (aa', bb') = (A'A, B'B)$$

Hence $$(AA', BB') = (A'A, B'B)$$

In this projectivity, the corresponding points correspond to one another doubly. Since this is true for any pair of conjugate points we conclude that *the pairs of conjugate points with respect to a conic on a line, which is not a tangent to the conic, are pairs of corresponding points of an involution.*

If the line g meets the conic in two points E, F, every pair of conjugate points are harmonically separated by E, F. So, E, F are the double points and the involution is hyperbolic. If the line g does not meet the conic, there are no (real) double points and the involution is elliptic.

In a similar manner, since $(aa', bb') = (a'a, b'b)$, *the pairs of conjugate lines with respect to a conic, through a point not on the conic, are pairs of corresponding lines of an involution.* If the point is outside the conic, the two tangents to the conic drawn from the point are the double lines of the involution and the involution is hyperbolic. If the point lies inside the conic, the involution is elliptic.

CHAPTER IX

GEOMETRY IN THE EXTENDED CARTESIAN PLANE

28. Points and line at infinity. We have up till now been dealing with geometry in the ordinary Euclidean plane where there is the basic axiom of Euclidean parallelism. In § § 7.1 and 9 we have noticed that if in the cross-ratio $(AB, CD) = a : b$ three of the four collinear points and the ratio $a : b$ are arbitrarily given, then, with the exception of a special case, the remaining point can be determined. In § 13 it has been noticed that the polar of a point with respect to a central conic exists, unless the point is the centre of the conic. And in § 27 it has been seen that in a given projectivity between two rows, the point corresponding to a given point can be determined, unless the given point is one of the particular points considered there or the centre of an involution.

In order to avoid these exceptional cases and to make our geometry more consistent and useful, we *extend* the Euclidean plane by introducing new elements which are merely our assumptions. These new elements are points and a straight line, and they are called the *points at infinity* (or, the ideal points) and the *line at infinity* (or, the ideal line). We extend every row of points by *one* point at infinity, the points at infinity of different rows being identical if and only if the bases of these rows are parallel lines. Furthermore, we assume that the line at infinity passes through every point at infinity and through no other point. The plane thus extended shall be called the *extended Cartesian plane*, or simply, the extended plane. In future, the points and lines of the nonextended Euclidean plane shall be called *ordinary* points and lines. Thus 'point' and 'line' in the extended plane may be either ordinary or at infinity.

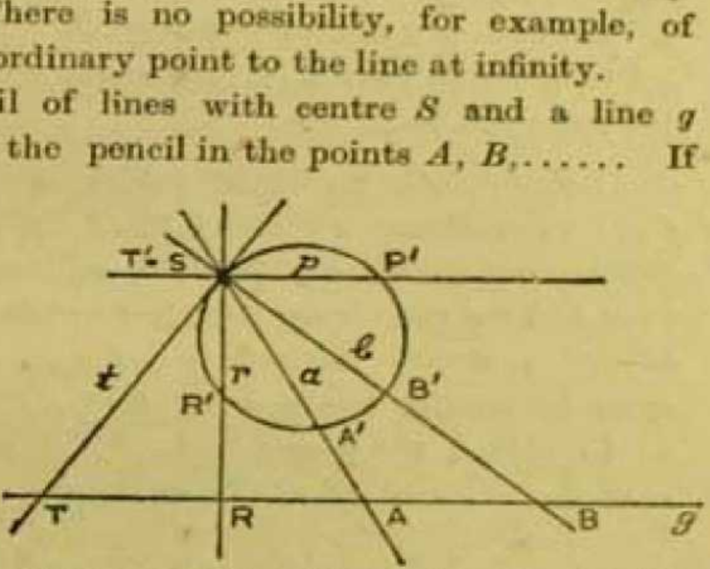
The pencil of lines passing through a point at infinity P consists therefore of a pencil of ordinary parallel lines together with the line at infinity. Thus there exists a one-to-one correspondence between the pencils Π of ordinary parallel lines and the points at infinity P. The line joining an ordinary point P_1 with P is the line of the pencil Π passing through P_1 and obviously there exists one and only one such line. Two ordinary points are joined in the extended plane by a line in the same manner as they are joined in the nonextended plane, and two points at infinity are joined only by the line at infinity. Thus, the extended plane has the following two properties, the first of which holds also in the Euclidean plane, namely, *Two distinct points are joined by one and only one line.*

Two distinct lines intersect in one and only one point.

For, if the two lines are intersecting in the Euclidean plane, they intersect in no other ordinary point and in no point at infinity, as the points of infinity of these lines are different ; if the lines are parallel, they intersect in one point at infinity and in no ordinary point ; and if one of the lines is the line at infinity, they intersect in the only point at infinity of the other line.

The above pair of properties have their implications that we have to expect by making the extension we have made. Certain notions of the Euclidean plane cease to be applicable unreservedly in the extended plane. As for example, the *distance* between two points exists only if both the points are ordinary ; the *angle* between two lines exists only if both the lines are ordinary. There is no possibility, for example, of drawing a perpendicular from an ordinary point to the line at infinity.

As in § 7.1, consider a pencil of lines with centre S and a line g meeting the lines a, b,......of the pencil in the points A, B,...... If the line p of the pencil is parallel to g, then p and g meet in a point at infinity P.



In order to understand the structure of the line, suppose that a line r of the pencil (S) starting from any definite position rotates about S in any sense through an angle 2π, thus describing the whole of the pencil. What happens to the corresponding point R on g ? It moves on g in a certain direction taking up different positions until it coincides with the point at infinity P and then it appears on the other side of the starting position and, moving in the same direction as before, comes back to the original position. Thus, as the line r describes the whole pencil, the point R describes the whole line g, So, *a line is to be regarded as a closed figure.* To assist the idea, let a circle be drawn through S and let the lines a, b, p....of the pencil meet the circle in A', B', P'...... If the tangent t to the circle at S meet g in T, we denote the point S by the letter T'. So, to every point R of g corresponds a point R' of the circle, and conversely. It can now be seen that as R' describes the whole circle, the point R describes the whole line.

29. Projective pencils and rows. An ordinary point can be taken as the centre of a pencil of lines and a pencil of parallel lines has its centre in a point at infinity, the line at infinity being a line of this pencil. So, every pencil has a centre. The set of all points (including the point at infinity) on a line shall now form a row. So, every line (including the line at infinity) can now be taken as the base of a row. Hence, a row can be obtained as the section of a pencil by a line and a pencil can be obtained as a projection of a row from an external point. The rows which are the sections of one and the same pencil are *perspective* and two perspective rows have a *self-corresponding* point, namely the point where their bases meet. Two pencils which are the projections of one and the same row are perspective and two perspective pencils have a self-corresponding line, namely the line joining their centres. Conversely, if two rows or two pencils have a self-corresponding element, they are perspective. *A row and a pencil are perspective* when the pencil is a projection of the row or the row is a section of the pencil. In this case, a point of the row and a line of the pencil are corresponding elements if they are *conjoint*, i.e., if the point lies on the line.

Let $(A_iB_iC_i\ldots\ldots)$ or (s_i) denote rows whose points are A_i, B_i, C_i,......on bases s_i, $i=1,2,3,\ldots\ldots$Similarly, Let $(a_ib_ic_i\ldots\ldots)$ or (S_i) denote pencils whose lines are a_i, b_i, c_i......through centres S_i. The rows and the pencils are obtained as follows: Project the row $(A_1B_1C_1\ldots)$ so as to obtain the pencil $(a_1b_1c_1\ldots)$; take the section of $(a_1b_1c_1\ldots)$ so as to obtain the row $(A_2B_2C_2\ldots)$; again project $(A_2B_2C_2\ldots)$ so as to obtain the pencil $(a_2b_2c_2\ldots)$ and so on. In this way we obtain a sequence of alternate rows and pencils.

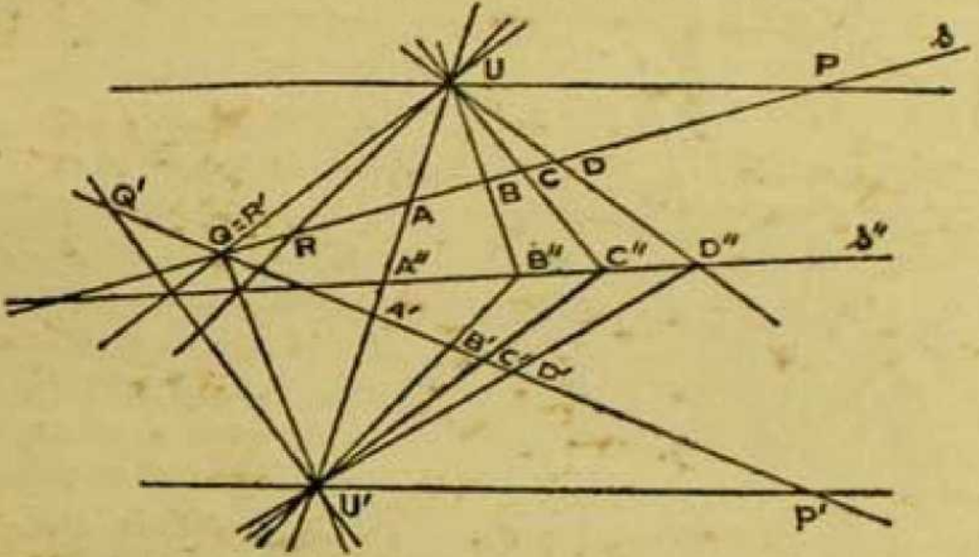
$$(s_1),\ (S_1),\ (s_2),\ (S_2),\ (s_3),\ (S_3)\ldots\ldots\ldots\ldots,$$

any two consecutive forms being perspective. In this chain of projections and sections, to any element of one form corresponds a definite element of another form. For example, to the point A_p of (s_p) corresponds the line a_q of (S_q) and also the point A_r of (s_r). Since cross-ratio remains invariant by projection or section (§ 7.1), any two forms in the above chain are projective (§ 25) and the corresponding elements in them are determined as stated. In the above chain, the points S_i and the lines s_i need not be all different. That is, we may have projective cobasal rows and concentric pencils.

Given three elements of one form to correspond respectively to three elements of another form, we can determine the projectivity between the two forms by the following geometrical constructions :

(1) Given three points A, B, C on a line s to correspond respectively to the three points A', B', C' on a line s', s and s' being distinct.

Take two points U, U' on the line joining any pair of the given corresponding points, say A, A'; let the lines UB, $U'B'$ meet in B'' and UC, $U'C'$ meet in C''; join B'', C'' by a line s'' to meet the line UU' in A''. Then the rows $(ABC\ldots.)$ and $(A'B'C'\ldots.)$ are each perspective to the row $(A''B''C''\ldots.)$ and are therefore projective to one another. To find the point D' of (s') which corresponds to a point D of (s), we join U and D to meet s'' in D''; then D' is the point of intersection of the lines $U'D''$ and s'. If D is the point at infinity on s, we have to draw the parallel to s through U to meet s'' in D'' and then obtain D' as before. To the point $Q \equiv R'$ in which the lines s and s' meet corresponds the points Q' and R obtained by the same construction.

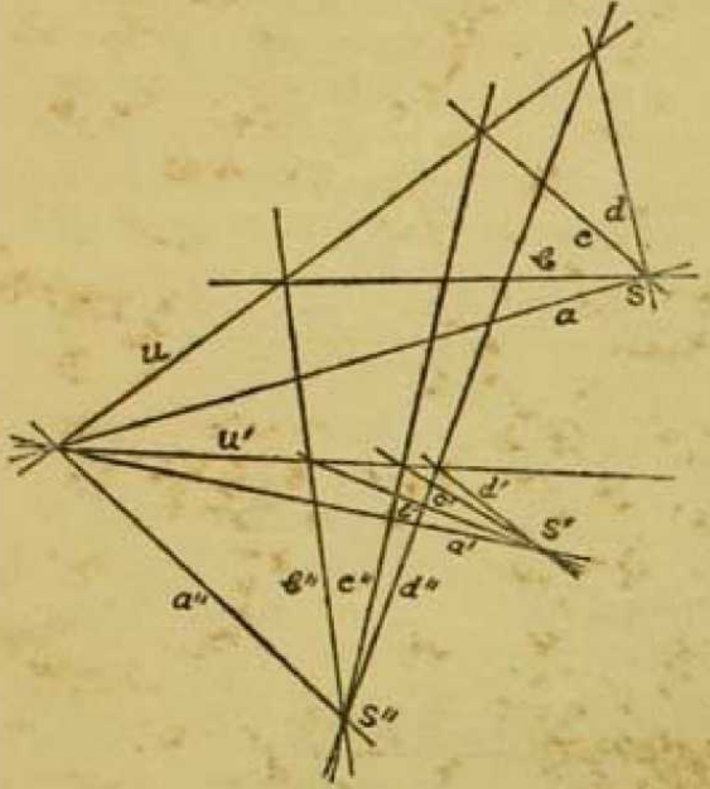


In the perspectivity between the rows (s) and (s''), if the point P of (s) be such that the line UP is parallel to s'', then the point P'', which corresponds to P, is the point at infinity on s''. In any perspectivity between two rows there are, in general, two such points P, one belonging to each row.

(2) Given three lines a, b, c through a point S to correspond respectively to the three lines a', b', c' through a point S', S and S' being distinct: In what follows it will be convenient to denote the point of intersection of two lines, p and q say, by the notation pq.

Take two lines u, u' through the point of intersection of any pair of the given corresponding lines, say a, a'; let the points bu and $b'u'$ lie on the line b'' and let the points cu and $c'u'$ lie on the line c''; let b'', c'' meet in S'' and let a'' be the line joining S'' and aa'. Then the pencils $(abc\ldots.)$ and $(a'b'c'\ldots.)$ are each perspective to the pencil $(a''b''c''\ldots.)$ and are therefore projective to one another. To find the line d' of (S') which corresponds to a line d of (S), we join S'' and du by the line d''; then d'

is the line joining S' and $u'd''$. The line SS' belongs to both (S) and (S'); when this line is regarded as belonging to one, the corresponding line in the other pencil is also found by the above construction. The above construction also holds if either S or S' or both are points at infinity. If S, for example, is a point at infinity, a, b, c, d are parallel.



(3) Given two triads of points A, B, C, and A', B', C' on the same line s.

In this case we project one of the triads A', B', C' from an external point so as to obtain a triad of lines a', b', c' and take the section of these lines by a transversal s' so as to obtain another triad of points A'', B'', C''. Then $(A'B'C'\ldots.)$ and $(A''B''C''\ldots.)$ are perspective. Now $(ABC\ldots.)$ and $(A''B''C''\ldots.)$ on the bases s and s' can be made projective by construction (1) given before. Therefore, we obtain a construction for the projectivity between $(ABC\ldots.)$ and $(A'B'C'\ldots.)$.

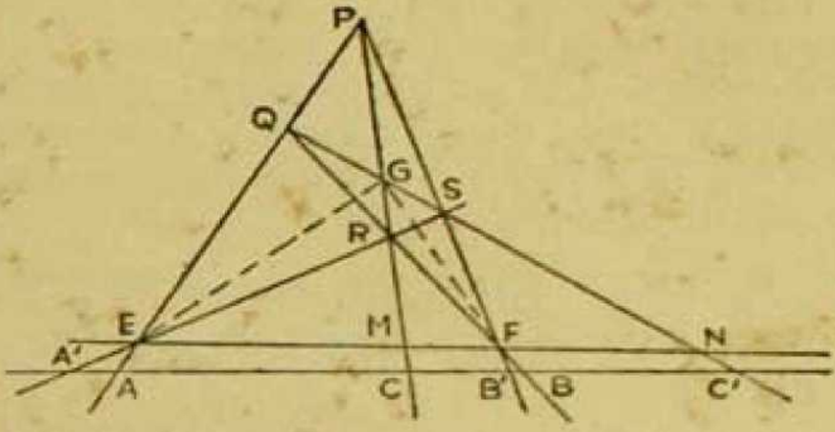
(4) Given two triads of lines a, b, c and a', b', c' concurring in the same point S:

As in (3), we construct a pencil $(a''b''c''\ldots.)$ with a centre S' perspective to $(a'b'c'\ldots.)$. Then, by the construction (2), we make $(abc\ldots.)$ and $(a''b''c''\ldots.)$ projective. Thus, we obtain a construction for the projectivity between $(abc\ldots.)$ and $(a'b'c'\ldots.)$.

As an *application*, let us show by geometrical construction that $(AB, CD)=(BA, DC)$ as seen in § 8.

Draw a line through D and project from an external point M the points A, B, C, D on this line so as to obtain the four points E, F, G, D. Let the lines AF and MC meet in N. Then, projecting from F on the line MN, we have $(AB, CD)=(NM, CG)$; similarly, projecting from A, $(NM, CG)=(FE, DG)$; and finally projecting from M, $(FE, DG)=(BA, DC)$. Thus $(AB, CD)=(BA, DC)$. Similarly, it can be shown that $(AB, CD)=(CD, AB)=(DC, BA)$.

29.1. Complete quadrangle and complete quadrilateral. Four points P, Q, R, S, no three of which are collinear, generate a figure called a *complete quadrangle.* The four points are called the *vertices* and the six lines PQ, PR, PS, QR, QS, RS joining the vertices in pairs are the *sides* of the complete quadrangle. Two sides which do not meet in a vertex are said to be *opposite*; accordingly, there are three pairs of opposite sides PQ, RS; PS, QR; PR, QS. The three points E, F, G in which the opposite sides intersect in pairs are called the *diagonal points* and the triangle EFG is called the *diagonal triangle* of the complete quadrangle.



Before deducing the properties of a complete quadrangle we first notice that if $(AB, CD)=(CB, AD)$, then $(AC, BD)=-1$. For, the equality of the cross-ratios gives

$$\frac{\overline{CA}}{\overline{CB}}\,\frac{\overline{DB}}{\overline{DA}}=\frac{\overline{AC}}{\overline{AB}}\,\frac{\overline{DB}}{\overline{DC}}, \text{ or } \frac{\overline{AB}}{\overline{CB}}\Big/\frac{\overline{DA}}{\overline{DC}}=(AC, BD)=-1$$

Now, let the lines PR, QS meet the line EF in the points M, N. Projecting from R on the line QS we obtain $(EM, FN)=(SG, QN)$; and projecting from P on the line EF, $(SG, QN)=(FM, EN)$. Accordingly, $(EM, FN)=(FM, EN)$. Therefore $(EF, MN)=-1$; that is, the four points E, F, M, N are harmonic points. Thus, *if two opposite sides of a complete quadrangle meet in a point E, two other opposite sides meet in a point F and the two remaining sides meet the line* EF in M, N, *then E, F separate M, N harmonically.* In the same way we obtain four harmonic points on each of the other two sides FG, GE of the diagonal triangle. From this property we obtain a geometrical construction for determining the fourth point of four harmonic points when three of the points are given. For, let E, M, F be given. To find N such that $(EF, MN)=-1$, we, draw through E any two lines EP and ER and through M we draw any line meeting EP and ER in P and R respectively; join FP and FR to meet ER and EP in S and Q respectively; then N is determined as the intersection of the lines QS and EF. It is evident that in whatever way this construction is made, the point N remains fixed, because the cross-ratio is constant.

Of the three given points E, M, F let M lie midway between E, F. We can take any triangle PEF and take the points Q, S as the middle points of the segments PE, PF. So, the remaining point R of the complete quadrangle is the median point of the triangle PEF. Then, since the lines QS and EF are parallel, the point N is the point at infinity on EF. Thus, *the middle point of a segment EF is the harmonic conjugate of the point at infinity of the line EF with respect to E, F.* In this case, since the cross-ratio

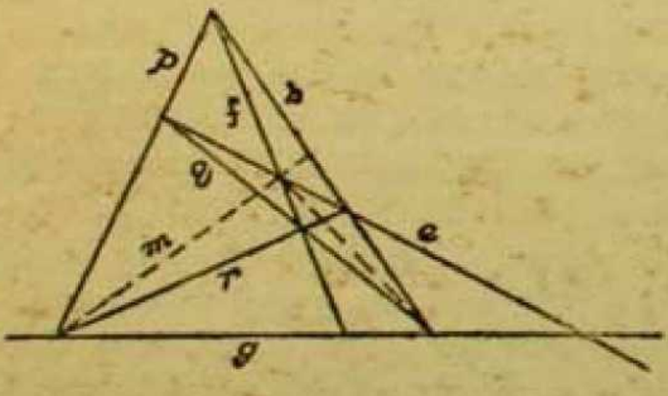
$$(EF, MN) = -1, \text{ so } \overline{NE}/\overline{NF} = 1$$

Thus, *if E, F are two ordinary points and N the point at infinity of the line EF, then*

$$\overline{NE}/\overline{NF} = 1 \tag{9.0}$$

Again, it is obvious that the lines GM and GN are harmonically separated by the lines GE and GF. Similarly, there are four harmonic lines through each of the other two diagonal points E, F. Thus, *in a complete quadrangle, if E, F, G are the diagonal points, the two sides of the quadrangle passing through any one of the diagonal points E are harmonically separated by the line EF, EG.*

Four lines p, q, r, s, no three of which are concurrent, generate a figure called a *complete quadrilateral.* The four lines are called the *sides* and the six points pq, pr, ps, qr, qs, rs, which are the intersections of the sides in pairs are called the *vertices of* the complete quadrilateral. Two vertices which do not lie on a side are said to be *opposite.* Accordingly, there are three pairs of opposite vertices pq, rs; ps, qr; pr, qs. The three lines e, f, g which join pairs of opposite vertices are called the *diagonals* and the triangle formed by e, f, g the *diagonal triangle* of the complete quadrilateral.



We can deduce the harmonic properties of a complete quadrilateral in the same way as obtained for a complete quadrangle. We only formulate the results : *If two opposite vertices of a complete quadrilateral lie on a line e, two other opposite vertices lie on a line f and the two remaining vertices are joined to the point ef by the lines m, n, then* $(ef, mn) = -1$. From this property we obtain a geometrical construction for determining the fourth line of four harmonic lines when three of the lines are given.

Again, *in a complete quadrilateral, if e, f, g are the diagonals, then the two vertices of the quadrilateral lying on any one of the diagonals e are harmonically separated by the points ef, eg.*

Consider again the complete quadrangle $PQRS$ (See Fig). Let the pairs of opposite sides PQ, RS ; QR, PS ; PR, QS be cut by a transversal in pairs of points (A, A'), (B, B'), (C, C'). Projecting from R on the line QS, we obtain the cross-ratios $(A'C, BC') = (SG, QC')$; and projecting from P on the line AA', $(SG, QC') = (B'C, AC')$. But $(B'C, AC') = (AC', B'C)$. Therefore $(A'C, BC') = (AC', B'C)$. In this projective correspondence of points, the points C, C' correspond to one another doubly. Hence, by § 27, (A, A'), (B, B'), (C, C') are pairs of corresponding points of an involution.

Thus, *a transversal cuts the three pairs of opposite sides of a complete quadrangle in three pairs of points of an involution.*

From this property we obtain a geometrical construction for determining the point C' which corresponds to a point C in an involution defined by two pairs of corresponding points (A, A'), (B, B') [one construction has been given in application (1) § 27]. For, draw any two lines through A, A' and draw any line through C to meet them in P and R ; let BR and $B'P$ meet AP and $A'R$ in Q and S respectively. Then the point C' is given as the intersection of the lines QS and AB. It may be of interest to see how by taking different positions of the transversal AB we obtain hyperbolic and elliptic involutions when the complete quadrangle is given. Let AB be parallel to QS. Then with the same letters as before, we have from similar triangles,

$$\frac{\overline{CA'}}{\overline{CR}} = \frac{\overline{GS}}{\overline{GR}}, \quad \frac{\overline{CA}}{\overline{CP}} = \frac{\overline{GQ}}{\overline{GP}}, \quad \frac{\overline{CB}}{\overline{CR}} = \frac{\overline{GQ}}{\overline{GR}}, \quad \frac{\overline{CB'}}{\overline{CP}} = \frac{\overline{GS}}{\overline{GP}}.$$

It follows

$$\overline{CA}.\overline{CA'} = \overline{CB}.\overline{CB'}$$

Hence C is the centre of the involution (§ 27). But since AB is parallel to QS, C corresponds to the point at infinity of AB. Thus, *the centre of an involution of points on a line corresponds to the point at infinity of the line.*

We can deduce similar involutory properties of a complete quadrilateral : *The lines which join any point with the three pairs of opposite vertices of a complete quadrilateral form three pairs of corresponding lines of an involution.* From this property we obtain a method of constructing geometrically the line c' which corresponds to a line c in an involution defined by two pairs of corresponding lines (a, a'), (b, b').

30. Homogeneous coordinates of a point. In order to take advantage of the points at infinity analytically, we introduce a new system of coordinates. A pair of numbers (x, y) has already been used to represent the (orthogonal) Cartesian coordinates of an ordinary point P. Let (x_1, x_2, x_3) be any three real numbers such that

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}. \tag{9.1}$$

or,

$$x : y : 1 = x_1 : x_2 : x_3$$

We shall then say that the ordered triad of numbers (x_1, x_2, x_3) are the *homogeneous Cartesian coordinates* of P. We say homogeneous, because it transforms any integral algebraic equation in x, y into a homogeneous integral algebraic equation in x_1, x_2, x_3, the degree of the equation remaining unchanged ; for example, a linear equation in nonhomogeneous coordinates $ax + by + c = 0$ is transformed in homogeneous coordinates into

$$ax_1 + bx_2 + cx_3 = 0 \tag{9.2}$$

Thus an ordinary point is represented by three coordinates (x_1, x_2, x_3), provided that the last coordinate x_3 does not vanish. It also follows from (9.1) that we require only the ratio of these coordinates ; that is to say, (x_1, x_2, x_3) and $(\rho x_1, \rho x_2, \rho x_3)$, where ρ is any arbitrary constant not equal to zero, represent the same point. We shall express this by writing $(x_1, x_2, x_3) = \rho(x_1, x_2, x_3)$. We shall also, for the sake of brevity, denote the coordinates (x_1, x_2, x_3) by (x_i), $i = 1, 2, 3$.

Now, a point lies on a line if its coordinates satisfy the equation of the line. If $\rho(x_1, x_2, x_3)$ satisfies (9.2), then three cases have to be considered :

(1) $x_1 = x_2 = x_3 = 0$. This solution is common to all the linear homogeneous equations and has therefore no geometrical importance.

(2) $x_3 \neq 0$. These solutions furnish, by (9.1), ordinary points (x, y) situated on the ordinary line $a_1 x + b_1 y + c = 0$.

(3) $\rho(b, -a, 0)$. This solution is independent of c, and is therefore a common solution of the pencil of lines parallel to (9.2). On the other hand, this pencil is determined by the ratio $a : b$. Hence we *define* $\rho(b, -a, 0)$ as the homogeneous coordinates of the point at infinity in which the parallel lines of the pencil $ax + by + c = 0$ (c arbitrary) meet. The points at infinity satisfy therefore the condition $x_3 = 0$. Hence (9.2) denotes a line of the extended plane also if $a = b = 0, c \neq 0$. As every point of the extended plane has coordinates and every $\rho(x_1, x_2, x_3)$ denotes a point, except for $x_1 = x_2 = x_3 = 0$, so every equation (9.2) denotes a line if every coefficient is not zero. Given the coordinates $\rho(a_1, a_2, a_3)$ of a particular

point, then *either* $a_3 \neq 0$ and the point is the ordinary point $x = a_1 : a_3$, $y = a_2 : a_3$ *or* $a_3 = 0$ and the point is the point at infinity of the pencil of parallel lines l for which

$$\cos(x, l) = \pm a_2 / \sqrt{(a_1^2 + a_2^2)}$$
$$\cos(y, l) = \mp a_1 / \sqrt{(a_1^2 + a_2^2)}$$

Let (a_1, a_2, a_3), (b_1, b_2, b_3) be two distinct points. If the equation of the line joining the two points is (9.2), we must have

$$aa_1 + ba_2 + ca_3 = 0, \quad ab_1 + bb_2 + cb_3 = 0$$

Hence, the equation of the line, obtained by eliminating a, b, c between the three equations is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

As adopted in § 23, let the determinant on the left-hand side be denoted by $| x\, a\, b |$. If (c_1, c_2, c_3) is a third point collinear with the two given points, we must have $| a\, b\, c | = 0$. Similarly, if three distinct lines

$$a_ix_1 + b_ix_2 + c_ix_{3=0}, \quad i = 1.\ 2, 3$$

are concurrent, we must have $| a\, b\, c | = 0$.

30.1 Linear dependence of points and lines. In nonhomogeneous coordinates, if (a, a'), (b, b') are two points, then the row of points on the line joining the two points is given by (§ 6)

$$\left.\begin{aligned} x &= \gamma a + \lambda b \\ y &= \gamma a' + \lambda b' \end{aligned}\right. \quad \gamma + \lambda = 1$$

Now, from (9.1), $\qquad x_1 : x_2 : x_3 = x : y : 1$

And if $\qquad a : a' : 1 = a_1 : a_2 : a_3, \quad b : b' : 1 = b_1 : b_2 : b_3,$

then putting $\gamma / a_3 = \gamma'$, $\lambda / b_3 = \lambda'$

$$(\gamma a + \lambda b) : (\gamma a' + \lambda b') : (\gamma + \lambda) = (\gamma' a_1 + \lambda' b_1) : (\gamma' a_2 + \lambda' b_2) : (\gamma' a_3 + \lambda' b_3)$$

Therefore, the homogeneous representation of the row joining the two points (a_1, a_2, a_3) and (b_1, b_2, b_3) is given by

$$\begin{aligned} \rho x_1 &= \mu a_1 + \nu b_1 \\ \rho x_2 &= \mu a_2 + \nu b_2 \qquad (9.3) \\ \rho x_3 &= \mu a_3 + \nu b_3 \end{aligned}$$

where μ, ν are any two arbitrary quantities. The points of the row will be obtained by pairs of values given to μ, ν other than $(\mu, \nu) = (0, 0)$. The values of μ, ν which make $x_3 = 0$ give the point at infinity of the row.

Thus, we may say that any point (x_i) of a row can be expressed as a *linear combination* of two other points (a_i), (b_i) of the row.

Let $Q_1 = (a_i)$, $Q_2 = (b_i)$, $Q = (p_i)$, $Q' = (q_i)$ be four collinear points. So we can write

$$p_i = \mu a_i + \nu b_i, \quad q_i = \mu' a_i + \nu' b_i.$$

Then, as in § 6, the cross-ratio

$$(Q_1 Q_2, QQ') = \nu\mu' / \nu'\mu.$$

Let m, n be any two of the numbers 1, 2, 3, $m \neq n$. Then

$$\begin{vmatrix} p_m & p_n \\ b_m & b_n \end{vmatrix} = \begin{vmatrix} \mu a_m + \nu b_m & \mu a_n + \nu b_n \\ b_m & b_n \end{vmatrix} = \mu \begin{vmatrix} a_m & a_n \\ b_m & b_n \end{vmatrix}$$

Therefore $$\mu = (p_m b_n - b_m p_n)/(a_m b_n - b_m a_n)$$

Similarly $$\nu = (p_m a_n - a_m p_n)/(b_m a_n - a_m b_n)$$

$$\mu' = (q_m b_n - b_m q_n)/(a_m b_n - b_m a_n), \quad \nu' = (q_m a_n - a_m q_n)/(b_m a_n - a_m b_n)$$

Hence, as in § 7, the cross-ratio

$$(Q_1 Q_2, QQ') = \frac{\nu\mu'}{\nu'\mu} = \frac{(p_m a_n - a_m p_n)(q_m b_n - b_m q_n)}{(p_m b_n - b_m p_n)(q_m a_n - a_m q_n)}.$$

This also shows that the cross-ratio is independent of m, n.

A number of points (a_i), (b_i), (c_i),..........are said to be *linearly dependent* if there exist quantities $p, q, r,$........., not all zero, such that the three equations

$$pa_i + qb_i + rc_i + \ldots\ldots = 0, \quad i = 1, 2, 3, \tag{9.4}$$

are satisfied. If the number of points be more than three, the points are always linearly dependent; for, we shall have three equations (9.4) in more than three unknowns $p, q, r,$........., and so solutions, other than $(0, 0, \ldots\ldots, 0)$ exist.

Consider the row (9.3). The equations can be written as

$$\mu a_i + \nu b_i - x_i = 0, \quad i = 1, 2, 3.$$

So, if (c_i) is a point of the row, we have

$$\mu a_i + \nu b_i - c_i = 0$$

This shows that *three collinear points are linearly dependent*. Conversely, *if three points are linearly dependent, they are collinear*. For, let (a_i) (b_i), (c_i) be the three points. If they are linearly dependent, three quantities p, q, r, not all zero, must exist such that the three equations

$$pa_i + qb_i + rc_i = 0$$

are satisfied. The condition for this is that the determinant $| \, a \; b \; c \, | = 0$. So, by the previous article, the three points are collinear. It can be easily seen that if two points are linearly dependent, they are identical.

If three points (a_i), (b_i), (c_i) are noncollinear, it is possible, for every triplet d_i, to solve the equations

$$pa_i + qb_i + rc_i = d_i$$

by suitable p, q, r. But (d_i) can be taken as the coordinates of a point. Thus, any arbitrary point of the plane can be expressed as a linear combination of three noncollinear points.

Consider two lines $\qquad \alpha \equiv a_1x_1 + a_2x_2 + a_3x_3 = 0$

and $\qquad \beta \equiv b_1x_1 + b_2x_2 + b_3x_3 = 0$

A linear combination of the two lines is a line

$$\mu\alpha + \nu\beta \equiv (\mu a_1 + \nu b_1)x_1 + (\mu a_2 + \nu b_2)x_2 + (\mu a_3 + \nu b_3)x_3 = 0$$

passing through the common point of $\alpha = 0$, $\beta = 0$. By giving all pairs of values to μ, ν, other than $(\mu, \nu) = (0, 0)$, in

$$\mu\alpha + \nu\beta = 0 \tag{9.5}$$

we obtain a pencil of lines.

A number of lines $\alpha = 0$, $\beta = 0$, $\gamma = 0, \ldots.$ are said to be *linearly dependent* if there exist quantities $p, q, r, \ldots.$, not all zero, such that

$$p\alpha + q\beta + r\gamma + \ldots. = 0 \tag{9.6}$$

vanishes identically, *i.e.*, vanishes for all sets of values of x_1, x_2, x_3. So, the coefficients of x_1, x_2, x_3 must separately vanish ; in other words if $\gamma \equiv c_1x_1 + c_2x_2 + c_3x_3 = 0, \ldots.$, then the three equations

$$pa_i + qb_i + rc_i + \ldots. = 0, \quad i = 1, 2, 3,$$

must be satisfied. It follows that if the number of lines be more than three, the lines are always linearly dependent.

Three lines are linearly dependent if and only if they are concurrent. For, if the lines $\alpha = 0$, $\beta = 0$, $\gamma = 0$ are linearly dependent, the three equations

$$pa_i + qb_i + rc_i = 0$$

must be satisfied for values of p, q, r other than all zero. The condition for this is that the determinant $| \, a \; b \; c \, | = 0$. So, by the last article, the three lines are concurrent. The converse is also true.

If two lines are linearly dependent, they are identical. Also, as before, any line of the plane can be expressed as a linear combination of three nonconcurrent lines.

As *applications* in the ordinary geometry, consider a triangle ABC. Let the origin be taken inside the triangle and let the equations of the lines BC, CA, AB in Hessian normal forms be

$$\alpha \equiv a_1x+a_2y+a_3 = 0, \quad \beta \equiv b_1x+b_2y+b_3 = 0, \quad \gamma \equiv c_1x+c_2y+c_3 = 0$$

Then the perpendicular distances of a point (x, y) on the sides

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0 \text{ are } \alpha, \beta, \gamma \text{ respectively.}$$

(1) The equations of the lines bisecting the angles of the triangle are $\beta-\gamma = 0$, $\gamma-\alpha = 0$, $\alpha-\beta = 0$. Since these lines are linearly dependent (as the sum of these functions vanishes identically), the bisector of the angles of a triangle are concurrent.

(2) Let D, E, F be the middle points of the sides BC, CA, AB respectively. The equation of the line AD is of the form $\mu\beta-\nu\gamma = 0$, or $\beta/\gamma = \nu/\mu$. The perpendicular distances of D from the sides CA, AB are halves of $c \sin C$, $c \sin B$ respectively, where c is the length of the side BC.

Therefore
$$\sin C/\sin B = \nu/\mu$$

Hence the equation of line AD is

$$\beta \sin B-\gamma \sin C = 0$$

Similarly the equations of BE, CF are

$$\gamma \sin C-\alpha \sin A = 0, \quad \alpha \sin A-\beta \sin B = 0$$

respectively. Since these lines are linearly dependent, the medians of a triangle are concurrent.

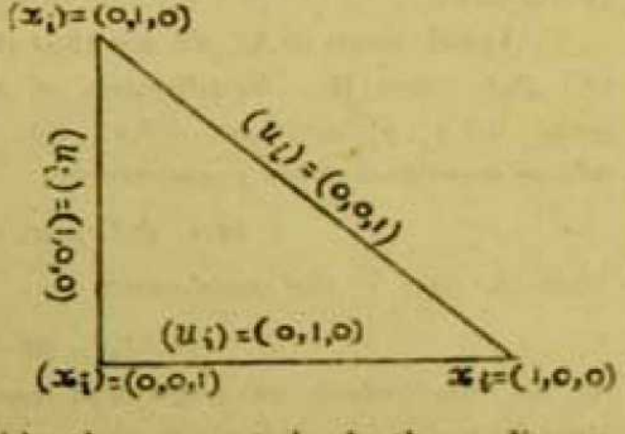
31. Homogeneous line coordinates. In the geometry which we have been studying, we have, up till now, regarded the point as the fundamental element. The line and the curves have been considered as the loci of points. So, we have begun with the coordinates of the point. On the other hand, there is no reason why the line should not be considered as the fundamental element. The point may then be defined as the intersection of lines and the curves as the envelopes of lines. For this purpose, we have to introduce the coordinates of the line. We have seen that every linear homogeneous equation in x_1, x_2, x_3

$$u_1x_1+u_2x_2+u_3x_3 = 0, \tag{9.7}$$

in which u_1, u_2, u_3 are not all zero, represents a line and conversely. If u_1, u_2, u_3 (or quantities proportional to them) are given, a line is fixed; the solutions of (9.7), other than (0, 0, 0), will then give the coordinates of those points which lie on the fixed line. On the other hand, if x_1, x_2, x_3 (or quantities proportional to them) are given, a point is fixed; (9.7) may then be considered as an equation in the variables u_1, u_2, u_3. Every solution of this equation, other than (0, 0, 0), may then be taken to represent a line which passes through the fixed point. Accordingly (u_1, u_2, u_3), where u_1, u_2, u_3 are not all zero, shall be regarded as the *homogeneous Cartesian coordinates of a line*. Thus, *when* x_1, x_2, x_3 *are the variables, the equation* (9.7) *is the equation of the line* (u_1, u_2, u_3) *in point coordinates* (x_1, x_2, x_3) ; *and when* u_1, u_2, u_3 *are the variables, then* (9.7) *is the equation of the point* (x_1, x_2, x_3) *in line coordinates* $(u_1, u_2, u_3,)$. It is evident that, as in the case of point coordinates, we are concerned only with the ratio of the line coordinates, so that (u_1, u_2, u_3) and $(\rho u_1, \rho u_2, \rho u\ )$ represent the same line. We express this by writing $(u_1, u_2, u_3) = \rho(u_1, u_2, u_3)$. We shall also denote the coordinates (u_1, u_2, u_3) of a line by the shorter notation (u_i).

From the definition of line coordinates it follows that *the condition that a point* (x_1, x_2, x_3) *and a line* (u_1, u_2, u_3) *are conjoint* (*i.e., the point lies on the line or the line passes through the point*) *is* $u_1x_1 + u_2x_2 + u_3x_3 = 0$.

Consider the equation (9.7) as the equation of a line (u_1, u_2, u_3) in point coordinates. The equation reduces to $x_3 = 0$ if $(u_1, u_2, u_3) = (0, 0, 1)$. So, the line at infinity has the coordinates (0, 0, 1). Similarly, the lines $x_2 = 0$ and $x_1 = 0$ have the coordinates (0, 1, 0) and (1, 0, 0) respectively. Again, if we regard (9.7) as the equation of a point (x_1, x_2, x_3) in line coordinates, the equation reduces to $u_3 = 0$ if $(x_1, x_2, x_3) = (0, 0, 1)$, reduces to $u_2 = 0$ if $(x_1, x_2, x_3) = (0, 1, 0)$, and to $u_1 = 0$ if $(x_1, x_2, x_3) = (1, 0, 0)$. Evidently, we get back the ordinary plane if we take away the line (0, 0, 1) or all points $(x_1, x_2, 0)$.



Let (u_i), (v_i) be two distinct lines. The point (x_i) common to the lines satisfies

$$u_1x_1 + u_2x_2 + u_3x_3 = 0$$
$$v_1x_1 + v_2x_2 + v_3x_3 = 0$$

Therefore solving the equations, we have, as coordinates,

$$(x_1, x_2, x_3) = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

This common point is a point at infinity if the last coordinate is zero, *i.e.*, if $(u_1, u_2, u_3) = (0, 0, u_3)$, or if $(v_1, v_2, v_3) = (0, 0, v_3)$, or if $u_1 : u_2 = v_1 : v_2$. So the two lines meet in a point at infinity if *either* one of the lines is the line at infinity *or* the two lines are parallel.

Similarly, let (x_i), (y_i) be two distinct points. The line (u_i) joining the points satisfies

$$u_1x_1 + u_2x_2 + u_3x_3 = 0$$

$$u_1y_1 + u_2y_2 + u_3y_3 = 0$$

Therefore, as before,

$$(u_1, u_2, u_3) = (x_2y_3 - y_2x_3,\ x_3y_1 - y_3x_1,\ x_1y_2 - y_1x_2)$$

Suppose that one of the given points is a point at infinity, $y_3 = 0$ say. Then

$$(u_1, u_2, u_3) = (-x_3y_2,\ x_3y_1,\ x_1y_2 - y_1x_2)$$

or

$$(u_1, u_2, u_3) = \left(-y_2, y_1, \frac{x_1y_2 - y_1x_2}{x_3}\right)$$

So, the coordinates of the line depend on the coordinates of the point at infinity.

Again, from (9.5), we see that if g_1, g_2 are two lines with coordinates (a_i), (b_i), then the coordinates of any line g passing through the common point of g_1, g_2 are $(\mu a_i + \nu b_i)$. So, take four concurrent lines g_1, g_2, g, g' whose coordinates are respectively

$$(a_i),\ (b_i),\ (\mu a_i + \nu b_i),\ (\mu' a_i + \nu' b_i)$$

Then, as in § 7, the cross-ratio

$$(g_1g_2, gg') = \nu\mu' / \nu'\mu$$

As an *application* of point and line coordinates, let us prove the theorem, already given in § 7.1, that the cross-ratio is unaltered by projection or section. Let

$$P_1 = (a_i), \quad P_2 = (b_i), \quad P = (\mu a_i + \nu b_i), \quad P' = (\mu' a_i + \nu' b_i)$$

be four collinear points. Then the cross-ratio

$$(P_1P_2, PP') = \nu\mu / \nu'\mu$$

Let $P_0 = (c_i)$ be an external point. Join P_0 with P_1, P_2, P, P' so as to obtain four concurrent lines p_1, p_2, p, p'. Then the line coordinates are as follows :

$$p_1 = \left(\begin{vmatrix} c_m & c_n \\ a_m & a_n \end{vmatrix}\right) \qquad p_2 = \left(\begin{vmatrix} c_m & c_n \\ b_m & b_n \end{vmatrix}\right),$$

$$p = \left(\begin{vmatrix} c_m & c_n \\ \mu a_m + \nu b_m & \mu a_n + \nu b_n \end{vmatrix} \right), \quad p' = \left(\begin{vmatrix} c_m & c_n \\ \mu' a_m + \nu' b_m, & \mu' a_n + \nu' b_n \end{vmatrix} \right)$$

where (m, n) are to be given the pairs of values (2,3), (3,1), (1,2). Or, supposing the coordinates of p_1, p_2 to be (d_i), (e_i) respectively, we may write

$$p_1 = (d_i),\ p_2 = (e_i),\ p = (\mu d_i + \nu e_i),\ p' = (\mu' d_i + \nu' e_i)$$

Therefore
$$(p_1, p_2, pp') = \nu\mu' / \nu'\mu$$

32. Principle of duality. Let us, for the moment, make no distinction between an ordinary point and a point at infinity, between an ordinary line and the line at infinity. That is to say, we suppose that there is nothing special about the points and the line at infinity and that all points stand on the same footing, so do all lines.

In the last article we have seen that we can regard either the point or the line as the fundamentel element of geometry; but we may also consider point and line as elements having equal right, these elements being connected by a certain *duality*. It is worthwhile to repeat some of the ideas discussed in the last three articles in the following manner:

To every point P there corresponds a triplet of numbers

$$(x_1, x_2, x_3) \neq (0, 0, 0) \tag{9.8}$$

determined up to a common arbitrary factor $\rho \neq 0$, and conversely. To every straight line p there corresponds a triplet of numbers

$$(u_1, u_2, u_3) \neq (0, 0, 0) \tag{9.9}$$

determined up to a common arbitrary factor $\sigma \neq 0$, and conversely. The last three numbers are the coefficients of the linear homogeneous equation

$$x_1 u_1 + x_2 u_2 + x_3 u_3 = 0 \tag{9.10}$$

representing the line p. The equation itself can be interpreted in this manner: *The point* (x_i) *and the line* (u_i) are *conjoint*. If a particular triplet (9.9) is given, the solutions (9.8) of (9.10) are the coordinates of the points lying on p. If, on the other hand, a particular triplet (9.8) is given, the solutions (9.9) of (9.10) are the coordinates of lines passing through p.

In the formulae (9.8), (9.9), (9.10) the notions of point and line are playing the same role, and if we interchange these two notions (or simply the letters x and u) the system of these formulae will not be altered. To every formula derived from a system of points and lines, there corresponds another formula which we get by interchanging x and u. Such pairs of formulae are said to be *dual*; and if the formulae are expressed as theorems, these theorems are also dual. In particular, the system composed of (9.8), (9.9), (9.10) is *self-dual*. For every theorem derived from a system which is dual in itself, we can therefore find another theorem which needs no new proof. Thus the ***principle of duality*** halves our labour. It is essential to know that the formulae of the ordinary (Euclidean) plane cannot be self-dual, because the coordinates of the points as well as of the lines are nonhomogeneous. It is only in the extended plane that the principle of duality can hold ; it cannot therefore be directly applied to theorems in which notions like "distance", "angle", "area", etc. occur, as these notions have a meaning only if we restrict our considerations to the nonextended plane. The cross-ratio, however, exists for every quadruplet of points of a row of the extended plane, the dual notion being the cross-ratios of four lines of a pencil.

Let

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

be a matrix of rank two ; then there exists one and only one solution (z_i) of the homogeneous equations

$$\sum_{i=1}^{3} a_i z_i = 0, \quad \sum_{i=1}^{3} b_i z_i = 0.$$

This algebraic result can be interpreted geometrically in two different ways : We may regard firstly (a_i), (b_i) as coordinates of two points and (z_i) as coordinates of the line joining them and secondly (a_i), (b_i) as coordinates of two lines and (z_i) as coordinates of their point of intersection. Thus we get the two dual theorems (already stated in § 28) :

Two different points are connected by one and only one line

Two different lines intersect in one and only one point.

In both cases the coordinates z_i are the minors of the above matrix.

Let now the three triplets (a_i), (b_i), (c_i) form a matrix of rank three. The matrix formed by the cofactors is also of rank three, as its determinant is the square of the determinant of the first matrix and is therefore not equal to zero. Thus the coordinates satisfying the above condition give rise to the two dual theorems :

If three points are noncollinear, the three lines joining them are nonconcurrent. If three lines are nonconcurrent, the three points of their intersections are noncollinear.

For the "translation" of a theorem into its dual theorem, the following "vocabulary" is helpful.

Point A	Line a
Line AB	Point ab
Collinear (noncollinear) points	Concurrent (nonconcurrent) lines
Cross-ratio of four collinear points.	Cross-ratio of four concurrent lines.

33. Loci and envelopes. The above list can be increased if we express geometrical entities of any kind by the help of homogeneous coordinates, and interchange the (x_i) and the (u_i) coordinates. Thus, let any curve of order n be expressed by

$$f(x, y) = 0 \tag{9.11}$$

where $f(x, y)$ is a polynomial of degree n. Put

$$x_3^n f\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right) = F(x_1, x_2, x_3)\,; \tag{9.12}$$

then (9.12) is a homogeneous polynomial of degree n. If we equate the polynomial to zero, namely

$$F(x_1, x_2, x_3) = 0, \tag{9.13}$$

then $F(\rho x_1, \rho x_2, \rho x_3) = 0$, and conversely. Hence the equation (9.13) is a condition to be satisfied by the points (x_1, x_2, x_3) of the extended plane. It is therefore a *locus of order n*. If (x, y) satisfies (9.11), then $\rho(x, y, 1)$ satisfies (9.13); but there may be also some points at infinity $(a, b, 0)$ which satisfy (9.13). Thus in deriving the homogeneous equation (9.13) from (9.11), some points at infinity may have been added to the curve (9.11) when it is changed to (9.13). These points cannot be expressed by nonhomogeneous coordinates.

The dual entity of the locus (9.13) is the *envelope of class n*

$$F(u_1, u_2, u_3) = 0 \tag{9.14}$$

formed by the lines (u_1, u_2, u_3) satisfying the condition (9.14).

In particular, consider a conic in the Euclidean plane, which is the locus of the points satisfying an equation the second degree

$$f(x, y) \equiv ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

Put
$$a = a_{11},\ c = a_{22},\ f = a_{33}$$
$$b = a_{12} = a_{21},\ d = a_{13} = a_{31},\ e = a_{23} = a_{32}$$

and introduce homogeneous coordinates; then we obtain a *locus of the second order*

$$F(x_1, x_2, x_3) \equiv \sum_{i,j} a_{ij}\, x_i x_j = 0, \quad i, j = 1, 2, 3 \tag{9.15}$$

To this locus corresponds dually an *envelope of the second class*

$$F(u_1, u_2, u_3) \equiv \sum_{i,j} a_{ij}u_iu_j = 0, \quad i, j = 1, 2, 3 \tag{9.16}$$

The geometrical connection between loci of second order and envelopes of second class will be discussed later on. In the meantime, suppose that (9.15) is a nondegenerate conic. Then it follows from (4.13), by introducing homogeneous coordinates, that the condition that two points (r_i) and (s_i) are conjugate with respect to (9.15) is

$$\sum_{i,j} a_{ij}r_is_j = 0 \tag{9.17}$$

and from (4.12) it follows that the polar of a point (r_i) with respect to (9.15) is given by

$$\sum_{i,j} a_{ij}r_ix_j = 0 \tag{9.18}$$

The equation (9.18) represents also the tangent to the conic at (r_i). These notions may be *dualised* : the condition for two conjugate lines and the equation of the pole of a line with respect to (9.16) are obtained from (9.17) and (9.18) by replacing the point coordinates by the line coordinates.

CHAPTER X

COLLINEATION AND CORRELATION

34. Transformation of collineation. The most general linear transformation of the homogeneous point coordinates is given by

$$\begin{aligned} \rho x_1' &= a_1x_1 + a_2x_2 + a_3x_3 \\ \rho x_2' &= b_1x_1 + b_2x_2 + b_3x_3 \\ \rho x_3' &= c_1x_1 + c_2x_2 + c_3x_3 \end{aligned} \tag{10.1}$$

where a_i, b_i, c_i are nine arbitrary constants and ρ is an arbitrary factor of proportionality. Let $|a\ b\ c|$ stand for the determinant of the coefficients and let A_i, B_i, C_i be the cofactors of a_i, b_i, c_i in $|a\ b\ c|$, $i = 1, 2, 3$. If $|a\ b\ c| = 0$, solutions for r_1, r_2, r_3, other than $(0, 0, 0)$, of the three equations

$$\sum_{i=1}^{3} a_i r_i = 0, \quad \sum_{i=1}^{3} b_i r_i = 0, \quad \sum_{i=1}^{3} c_i r_i = 0,$$

exist ; so, there are points (x_i) of which the transforms (x_i') are situated on a line $u_1x_1' + u_2x_2' + u_3x_3' = 0$. We shall always suppose that $|a\ b\ c| \neq 0$, so $\rho \neq 0$.

Multiplying the three equations (10.1) by A_i, B_i, C_i, we obtain the *inverse* transformation

$$\begin{aligned} \rho' x_1 &= A_1x_1' + B_1x_2' + C_1x_3' \\ \rho' x_2 &= A_2x_1' + B_2x_2' + C_2x_3' \\ \rho' x_3 &= A_3x_1' + B_3x_2' + C_3x_3' \end{aligned} \tag{10.1'}$$

The determinant $|A\ B\ C| \neq 0$, because $|a\ b\ c| \neq 0$.

Now consider all points lying on a line $u_1x_1 + u_2x_2 + u_3x_3 = 0$. It follows from the above inverse transformation that these points are transformed into points lying on a line $u_1'x_1' + u_2'x_2' + u'_3x_3' = 0$, where

$$\begin{aligned} \sigma u_1' &= A_1u_1 + A_2u_2 + A_3u_3 \\ \sigma u_2' &= B_1u_1 + B_2u_2 + B_3u_3 \\ \sigma u_3' &= C_1u_1 + C_2u_2 + C_3u_3 \end{aligned} \tag{10.2}$$

Since $|A\ B\ C| \neq 0$, (10.2) has its inverse

$$\begin{aligned} \sigma' u_1 &= a_1u_1' + b_1u_2' + c_1u_3' \\ \sigma' u_2 &= a_2u_1' + b_2u_2' + c_2u_3' \\ \sigma' u_3 &= a_3u_1' + b_3u_2' + c_3u_3' \end{aligned} \tag{10.2'}$$

The transformation (10.1) is a transformation of point coordinates which gives rise to the transformation (10.2) of line coordinates. We could have also started from (10.2) and obtained (10.1). A transformation of the type (10.1) or (10.2), for which the determinant of the coefficients is not zero, is called a *collineation* or a *projective transformation*. A collineation transforms, in general, a point at infinity into an ordinary point, the line at infinity into an ordinary line and *vice versa*. So, parallelism of lines is not, in general, preserved by collineation. Therefore, in a collineation, we do not recognise anything special about the points and the line at infinity. There is duality in collineation, because we have here transformations for both point and line coordinates; collinear points are transformed into collinear points and concurrent lines into concurrent lines. Equations (10.1) and (10.2) represent the same collineation. The inverse of a collineation is also a collineation.

Given the nine constants a_i, b_i, c_i, the collineation (10.1) is unique, *i.e.*, each point is transformed into a definite point, whatever arbitrary value the constant of proportionally ρ may have. The quantity ρ cannot be zero, but it should be noticed that it may have different values for different pairs of points. Thus, when $(p_i) \to (q_i)$, we may write

$$\rho_1 q_1 = \Sigma a_i p_i, \quad \rho_1 q_2 = \Sigma b_i p_i, \quad \rho_1 q_3 = \Sigma c_i p_i,$$

and when $(r_i) \to (s_i)$,

$$\rho_2 s_1 = \Sigma a_i r_i, \quad \rho_2 s_2 = \Sigma b_i r_i, \quad \rho_2 s_3 = \Sigma c_i r_i$$

The product of two collineations is a collineation. For, a collineation (10.1) transforming $(x_i) \to (x_i')$ followed by a collineation

$$\sigma x_1'' = \Sigma a_i' x_i', \quad \sigma x_2'' = \Sigma b_i' x_i', \quad \sigma x_3' = \Sigma c_i' x_i', \quad |a' \, b' \, c'| \neq 0,$$

transforming $(x_i') \to (x_i'')$ must be a transformation of the form

$$\sigma x_1'' = \Sigma a_i'' x_i, \quad \sigma x_2'' = \Sigma b_i'' x_i, \quad \sigma x_3'' = \Sigma c_i'' x_i,$$

where

$$|a'' \, b'' \, c''| = \frac{1}{\rho^3} |a \, b \, c| \, |a' \, b' \, c'| \neq 0$$

Therefore the product is a collineation. In order to reduce (10.1) in nonhomogeneous form, we may write it as

$$\frac{x_1'}{x_3'} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{c_1 x_1 + c_2 x_2 + c_3 x_3}, \quad \frac{x_2'}{x_3'} = \frac{b_1 x_1 + b_2 x_2 + b_3 x_3}{c_1 x_1 + c_2 x_2 + c_3 x_3}$$

and then apply (9.1) so as to obtain

$$x' = \frac{a_1 x + a_2 y + a_3}{c_1 x + c_2 y + c_3}, \quad y' = \frac{b_1 x + b_2 y + b_3}{c_1 x + c_2 y + c_3}$$

34.1. Properties of a collineation. Let us first prove the important property that *the cross-ratio is preserved by collineation.* Suppose that the four collinear points

$$P = (p_i), \quad Q = (q_i), \quad R = (\mu p_i + \nu q_i), \quad S = (\mu' p_i + \nu' q_i)$$

are transformed by (10.1) into the four points

$$P' = (p'_i), \quad Q' = (q'_i), \quad R' = (r'_i), \quad S' = (s'_i)$$

respectively. Then we must have

$$\rho_1 p_1' = \Sigma a_i p_i, \quad \rho_1 p_2' = \Sigma b_i p_i, \quad \rho_1 p_3' = \Sigma c_i p_i;$$

$$\rho_2 q_1' = \Sigma a_i q_i, \quad \rho_2 q_2' = \Sigma b_i q_i, \quad \rho_2 q_3' = \Sigma c_i q_i;$$

Therefore, as $\Sigma a_i(\mu p_i + \nu q_i) = \mu \Sigma a_i p_i + \nu \Sigma a_i q_i = \mu \rho_1 p_1' + \nu \rho_2 q_1'$,

$$\rho_3 r_1' = \mu \rho_1 p_1' + \nu \rho_2 q_1'$$

$$\rho_3 r_2' = \mu \rho_1 p_2' + \nu \rho_2 q_2'$$

$$\rho_3 r_3' = \mu \rho_1 p_3' + \nu \rho_2 q_3'$$

Similarly for $\quad s_1', s_2', s_3'$

Therefore, the coordinates of the four transformed points P', Q', R', S' are

$$(p_i'), \ (q_i'), \ \left(\mu \frac{\rho_1}{\rho_3} p_i' + \nu \frac{\rho_2}{\rho_3} q_i'\right), \ \left(\mu' \frac{\rho_1}{\rho_4} p_i' + \nu' \frac{\rho_2}{\rho_4} q_i'\right)$$

Hence, the cross-ratio

$$(PQ, RS) = \nu\mu'/\nu'\mu = (P'Q', R'S').$$

Obviously, the cross-ratio of four concurrent lines is also preserved. We may thus state that

A collineation of the plane establishes a one-to-one correspondence between the points, a one-to-one correspondence between the lines and preserves cross-ratio.

It should, however, be carefully understood that a collineation does not, in general, preserve angle, distance or the ratio of distances. The transformation of points into points or of lines into lines which is geometrically characterised by the operations, projections and sections (§ 29) is analytically expressed by the collineation. Hence a collineation is called a projective transformation. We next prove the following fundamental theorem.

Theorem. *There exists a unique collineation by which four given points forming a quadrangle are transformed into four other given points forming another quadrangle.*

We first consider a special case. Suppose that the four points

$$(1, 0, 0), \ (0, 1, 0), \ (0, 0, 1), \ (1, 1, 1).$$

between the lines of two pencils are called *one-dimensional transformations*. The transformations which establish a one-to-one correspondence between the points and between the lines of a plane field are called *two-dimensional transformations*, e.g., the transformations (10.1), (10.2).

If the coordinate system on a line, as given in § 1, be made homogeneous by writing $x = x_1 : x_2$, then a one-dimensional collineation or projective transformation of points is given by

$$\begin{aligned} \rho x_1' &= a_1 x_1 + a_2 x_2 \\ \rho x_2' &= b_1 x_1 + b_2 x_2 \end{aligned} \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0,$$

the points (x_1, x_2), (x_1', x_2') being corresponding points of two rows, either distinct or cobasal. The point at infinity $(1, 0)$ is transformed into the point (a_1, b_1). In non-homogeneous coordinates the above transformation is written as

$$x' = \frac{a_1 x + a_2}{b_1 x + b_2}$$

or

$$b_1 x x' - a_1 x + b_2 x' - a_2 = 0$$

If the two rows are cobasal, this collineation is an involution provided that $b_2 = -a_1$. So the involution is given by

$$x x' - \frac{a_1}{b_1}(x + x') - \frac{a_2}{b_1} = 0, \quad b_1 \neq 0$$

or

$$\left(x - \frac{a_1}{b_1}\right)\left(x' - \frac{a_1}{b_1}\right) = \frac{a_2}{b_1} + \frac{a_1^2}{b_1^2} = c \text{ (say)}$$

Changing the origin to a_1/b_1, the equation of the involution can be written as

$$x x' = c,$$

where the point with coordinate a_1/b_1 is the centre of the involution. (The results obtained here may be compared with those given in § 27).

One-dimensional affine transformation is obtained by specialising the above collineation, by putting $b_1 = 0$, as

$$\begin{aligned} \rho x_1' &= a_1 x_1 + a_2 x_2 \\ \rho x_2' &= \phantom{a_1 x_1 + {}} b_2 x_2 \end{aligned}$$

In nonhomogeneous coordinates, this affinity is

$$x' = \frac{a_1}{b_2} x + \frac{a_2}{b_2}$$

If the two rows are cobasal, this affinity is an *affine involution* provided that $b_2 = -a_1$. So, the involution is given by

$$x + x' = a_2/b_2$$

or

$$(x - a_2/2b_2) + (x' - a_2/2b_2) = 0$$

Changing the origin to $a_2/2b_2$, the equation of the affine involution takes the form

$$x + x' = 0$$

Therefore, if C is the point with the coordinate $a_2/2b_2$ and (P, P') a pair of corresponding points,

$$\overline{CP} + \overline{CP'} = 0.$$

So, C is the middle point of the segment PP'. Thus, *the affine involution of points is always hyperbolic, the double points being C and the point at infinity of the row.*

In the equation of a pencil of lines $\mu l_1 + \nu l_2 = 0$, (μ, ν) may be regarded as the homogeneous coordinates of a line of the pencil, because they specify the line. So, a one-dimensional collineation of lines is given by

$$\begin{aligned} \rho\mu' &= a_1\mu + a_2\nu \\ \rho\nu' &= b_1\mu + b_2\nu \end{aligned} \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0,$$

where (μ, ν), (μ', ν') are the coordinates of the corresponding lines of the two pencils. Finally, we may state the obvious theorem that *a projective correspondence between two one-dimensional geometric forms is uniquely determined by three distinct pairs of corresponding elements.*

36. Generalisation by collineation. A number of important generalisations of theorems in the extended Cartesian plane can be obtained by projective transformation, or, as we say, by projection.

Consider the triangle whose vertices A_1, A_2, A_3 have the coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and let A_4 be the point $(1, 1, 1)$. Take a point B, not on any side of the triangle, with coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. Let M_1 and B_1 be the points where the lines A_1A_4 and A_1B meet the line A_2A_3. Similarly, let M_2, B_2 be the points where the lines A_2A_4, A_2B meet A_3A_1 and M_3, B_3 the points where the lines A_3A_4, A_3B meet A_1A_2. Also, let N_1 be the harmonic conjugate of M_1 with respect to A_2, A_3; N_2 the harmonic conjugate of M_2 with respect to A_3, A_1; and N_3 the harmonic conjugate of M_3 with respect to A_1, A_2.

The equations of the sides A_2A_3, A_3A_1, A_1A_2 of the triangle are respectively $x_1 = 0$, $x_2 = 0$, $x_3 = 0$; the equations of the lines A_1A_4, A_2A_4, A_3A_4 are respectively

$$x_2 - x_3 = 0, \quad x_3 - x_1 = 0, \quad x_1 - x_2 = 0.$$

So, the coordinates of M_1 are $(0, 1, 1)$ and the coordinates of B_1 are $(0, \bar{x}_2, \bar{x}_3)$ (for, the equation of the line A_1B is $\bar{x}_3x_2 - \bar{x}_2x_3 = 0$; therefore $x_2/x_3 = \bar{x}_2/\bar{x}_3$; and B_1 lies on the lines A_1B and $x_1 = 0$).

Thus, if (p_i), (q_i), (r_i), (s_i) denote the coordinates of A_2, A_3, M_1, B_1 respectively, we have

$$r_i = p_i + q_i, \quad s_i = \bar{x}_2p_i + \bar{x}_3q_i, \quad i = 1, 2, 3$$

Therefore the cross-ratio

$$(A_2A_3, M_1B_1) = \bar{x}_2/\bar{x}_3.$$

Similarly

$$(A_3A_1, M_2B_2) = \bar{x}_3/\bar{x}_1 \quad \text{and} \quad (A_1A_2, M_3B_3) = \bar{x}_1/\bar{x}_2$$

Accordingly

$$(A_2A_3, M_1B_1)\,(A_3A_1, M_2B_2)\,(A_1A_2, M_3B_3) = 1 \qquad (10.5)$$

Again, by hypothesis,

$$(A_2A_3, M_1N_1) = (A_3A_1, M_2N_2) = (A_1A_2, M_3N_3) = -1$$

So, the coordinates of N_1, N_2, N_3 are respectively

$$(0, 1, -1), \quad (-1, 0, 1), \quad (1, -1, 0)$$

(for, if (r'_i) denote the coordinates of N_1, $r'_i = p_i - q_i$). Hence, *the three points* N_1, N_2, N_3 *are collinear and lie on the line* $x_1 + x_2 + x_3 = 0$

But from (8.1) we have the relations

$$(A_2A_3, B_1M_1)(A_2A_3, M_1N_1) = (A_2A_3, B_1N_1)$$

$$(A_3A_1, B_2M_2)(A_3A_1, M_2N_2) = (A_3A_1, B_2N_2)$$

$$(A_1A_2, B_3M_3)(A_1A_2, M_3N_3) = (A_1A_2, B_3N_3)$$

Therefore, multiplying these three relations we have from (10.5),

$$(A_2A_3, B_1N_1)(A_3A_1, B_2N_2)(A_1A_2, B_3N_3) = -1 \qquad (10.6)$$

Now, a collineation carries concurrent lines into concurrent lines, collinear points into collinear points, transforms the quadrangle $A_1A_2A_3A_4$ into another quadrangle and preserves cross-ratio. Therefore, applying a collineation, the results (10.5) and (10.6) may be generalised into the following theorems;

Theorem I. Let P_1, P_2, P_3 *be the vertices of a triangle and* Q *any point not lying on any side of the triangle. Also, let* P_1Q, P_2Q, P_3Q *meet the opposite sides of the triangle in* Q_1, Q_2, Q_3 *respectively. Then, if* R *is any other point not lying on any side of the triangle and if* P_1R, P_2R, P_3R *meet the opposite sides of the triangle in* R_1, R_2, R_3 *respectively,*

$$(P_2P_3,\ Q_1R_1)(P_3P_1,\ Q_2R_2)(P_1P_2,\ Q_3R_3) = 1.$$

Conversely, *if the product of the three cross-ratios is equal to unity, the lines* P_1R_1, P_2R_2, P_3R_3 *are concurrent.*

By dualising the above theorem we obtain the following :

Theorem I′. Let p_1, p_2, p_3 *be the sides of a triangle and* q *any line not passing through any vertex of the triangle. Also, let the lines joining the points* p_1q, p_2q, p_3q, *with the opposite vertices of the triangle be* q_1, q_2, q_3 *respectively. Then, if* r *is any other line not passing through any vertex of the triangle and if* r_1, r_2, r_3 *are the lines joining the points* p_1r, p_2r, p_3r *with the opposite vertices of the triangle,*

$$(p_2p_3,\ q_1r_1)(p_3p_1,\ q_2r_2)(p_1p_2,\ q_3r_3) = 1\ ;$$

and conversely, *if the product of the cross-ratios is equal to unity, the points* p_1r_1, p_2r_2, p_3r_3 *are collinear.*

Theorem II. Let P_1, P_2, P_3 *be the vertices of a triangle and* Q_1, Q_2, Q_3 *the points on the sides* P_2P_3, P_3P_1, P_1P_2 *respectively such that* P_1Q_1, P_2Q_2, P_3Q_3 *are concurrent in a point not lying on any side of the triangle. Then, if a transversal not passing through any vertex of the triangle meets the sides* P_2P_3, P_3P_1, P_1P_2 *in the points* R_1, R_2, R_3, *respectively,*

$$(P_2P_3,\ Q_1R_1)(P_3P_1,\ Q_2R_2)(P_1P_2,\ Q_3R_3) = -1.$$

Conversely, *if the product of the cross-ratios is equal to* -1, *then the points* R_1, R_2, R_3 *are collinear.*

By dualising the theorem II we obtain the following :

Theorem II′. Let p_1, p_2, p_3 *be the sides of a triangle and* q_1, q_2, q_3 *be the lines passing through the vertices* p_2p_3, p_3p_1, p_1p_2 *respectively such that* p_1q_1, p_2q_2, p_3q_3 *lie on a line not passing through any vertex of the triangle. Then, if a point not lying on any side of the triangle be joined to the vertices* p_2p_3, p_3p_1, p_1p_2 *by the lines* r_1, r_2, r_3 *respectively,*

$$(p_2p_3,\ q_1r_1)(p_3p_1,\ q_2r_2)(p_1p_2,\ q_3r_3) = -1\ ;$$

and conversely, *if the product of the cross-ratios is equal to* -1, *then the lines* r_1, r_2, r_3 *are concurrent.*

From the theorems I′ and II we obtain, by specialisation, two important theorems.

In theorem I′, let P_i be the vertex opposite to p_i, Q_i the point p_iq and R_i the point p_ir, $i = 1, 2, 3$. Then the result is

$$(P_2P_3, Q_1R_1)(P_3P_1, Q_2R_2)(P_1P_2, Q_3R_3) = 1.$$

Let us now suppose that one of the lines q, r, say r, is the line at infinity, then since (§ 29)

$$\frac{\overline{P_2R_1}}{\overline{P_3R_1}} = \frac{\overline{P_3R_2}}{\overline{P_1R_2}} = \frac{\overline{P_1R_3}}{\overline{P_2R_3}} = 1,$$

the above result becomes

$$\frac{\overline{P_2Q_1}}{\overline{P_3Q_1}} \frac{\overline{P_3Q_2}}{\overline{P_1Q_2}} \frac{\overline{P_1Q_3}}{\overline{P_2Q_3}} = 1.$$

We thus obtain the following theorem :

Theorem of MENELAUS. If a line not passing through any vertex of the triangle ABC meet the sides BC, CA, AB in A′, B′, C′ respectively, then

$$\frac{\overline{BA'}}{\overline{CA'}} \frac{\overline{CB'}}{\overline{AB'}} \frac{\overline{AC'}}{\overline{BC'}} = 1, \textit{ and conversely.}$$

Again, in theorem II, let us suppose that the transversal is the line at infinity. We have then the following theorem :

Theorem of CEVA. If A′, B′, C′ are points on the sides BC, CA, AB of a triangle ABC such that AA′, BB′, CC′, meet in a point not lying on any side of the triangle, then

$$\frac{\overline{BA'}}{\overline{CA'}} \frac{\overline{CB'}}{\overline{AB'}} \frac{\overline{AC'}}{\overline{BC'}} = -1, \textit{ and conversely.}$$

A number of theorems of the ordinary geometry are immediately seen to be particular cases of the above two theorems. Thus, the theorem that a line drawn parallel to one side of a triangle cuts the other two sides proportionately is a particular case of Menelaus' theorem (for, if the line is parallel to BC, $\overline{BA'}/\overline{CA'} = 1$). The theorems that the medians of a triangle are concurrent, the perpendiculars from the vertices on the opposite sides of a triangle are concurrent are particular cases of Ceva's theorem.

The theorem of proportion follows from the fact that the cross-ratio remains invariant by projection. In this case, there are two pencils of lines, the centre of one being a point at infinity.

DESARGUES' theorem on coplanar perspective triangles. If two triangles ABC and A′B′C′ are such that the lines AA′, BB′, CC′, joining the three pairs of vertices are concurrent, then the three pairs of corresponding sides BC, B′C′ ; CA, C′A′ ; AB, A′B′ meet in three collinear points. Conversely, if the three pairs of sides BC, B′C′ ; CA, C′A′ ; AB, A′B′ meet in three *collinear points, then the lines AA', BB', CC' joining the three pairs of corresponding vertices, are concurrent.*

Let the lines AA', BB', CC' meet in a point S, and let BC, $B'C'$ meet in L; CA, $C'A'$ meet in M; AB, $A'B'$ meet in N.

Suppose in the first place, that S is a point at infinity. If then two pairs of corresponding sides are parallel, the remaining two sides are also parallel. Hence L, M, N lie on the line at infinity and the theorem holds. Secondly, suppose that S is an ordinary point. If now two pairs of corresponding sides are parallel, the remaining two sides are also parallel, so that L, M, N lie on the line at infinity and the theorem holds.

Now, for the general case, we observe that parallelism of lines is not conserved by collineation. Therefore, the general theorem follows by applying a collineation.

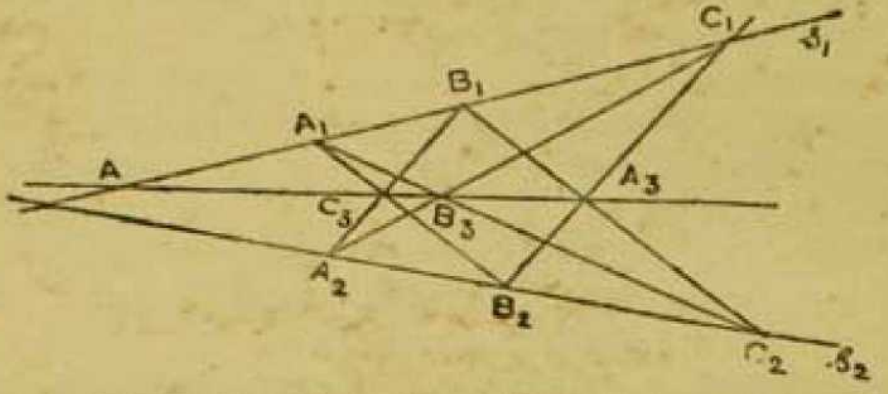
The converse theorem is easily proved by the indirect method. Let L, M, N be collinear and AA', BB' meet in S. Then if CC' does not pass through S, SC will cut $B'C'$ in some point D' distinct from C'. It follows from the general theorem just proved that the two triangles ABC and $A'B'D'$ are such that the pairs of corresponding sides meet in three collinear points. This means that $A'D'$, AC meet LN in the same point; but this is impossible unless D' coincides with C'.

The two triangles of Desargues' theorem are called two *perspective triangles*. The point S is called the *centre* and the line LMN the *axis* of perspectivity. The theorem and its converse are dual in the plane. A purely projective proof of the theorem will be given later.

Theorem of PAPPUS. If A_1, B_1, C_1 and A_2, B_2, C_2 are two triads of points on two lines and if the lines B_1C_2, C_1B_2 meet in A_3; C_1A_2, A_1C_2 meet in B_3; A_1B_2, B_1A_2 meet in C_3, then the three points A_3, B_3, C_3 are collinear.

Let A_1, B_1, C_1 lie on a line s_1 and A_2, B_2, C_2 lie on s_2. We shall, as before, consider initially the particular case where B_1C_2 is parallel to C_1B_2 and C_1A_2 parallel to A_1C_2. First suppose that s_1 and s_2 are parallel. Since B_1C_2, C_1B_2 are parallel and C_1A_2, A_1C_2 are parallel, so B_1A_2, A_1B_2 are also parallel. Hence A_3, B_3, C_3 lie on the line at infinity and the theorem holds. Secondly, suppose that s_1, s_2 meet in an ordinary point S. Here also A_3, B_3, C_3 are, as before, collinear and the theorem holds. From the particular case the general theorem follows by collineation. A purely projective proof of the theorem of Pappus is the following:

Let A_3B_3 meet s_1 in A and A_1B_2 in P. Then we are to show that P coincides with C_3. If C_1A_2 meet B_1C_2 in B, and C_1B_2 meet A_1C_2 in C, then, denoting projectivity by the symbol $\overline{\wedge}$,



$$C_1BB_3A_2 \,\overline{\wedge}\, C_1A_3CB_2, \text{ projecting from } C_2$$

and $$C_1A_3CB_2 \,\overline{\wedge}\, AA_3B_3P, \text{ projecting from } A_1$$

$$\therefore\ C_1BB_3A_2 \,\overline{\wedge}\, AA_3B_3P$$

In this projectivity, B_3 is self-corresponding ; so it must be a perspectivity (§ 29). Hence the lines C_1A, BA_3, A_2P, joining corresponding points, must be concurrent. But C_1A, BA_3 meet in B_1 ; so P lies on A_2B_1. Accordingly, P coincides with C_3.

Let us consider the configuration of Pappus formed by the nine points A_i, B_i, C_i, $i = 1, 2, 3$, and the nine lines each of which contains three of the points. The points on a line are either A_i, B_i, C_i or A_i, B_j, C_k, $i \neq j \neq k$, and through each point pass three of the lines. All the triangles contained in the figure may be arranged in different sets, each set containing three triangles, in the following way :

We first define inscribed and circumscribed triangles. A triangle Δ is said to be inscribed in a triangle Δ' when the vertices of Δ lie on the sides of Δ', one on each ; in this case we also say that Δ' is circumscribed to Δ. From this point of view, three triangles Δ_1, Δ_2, Δ_3 of the Pappus configuration form a set when Δ_3 is inscribed in Δ_2, Δ_2 is inscribed in Δ_1 and Δ_1 is inscribed in Δ_3. Take, for example, the triangle $A_1B_2C_2$. On the sides A_1B_2, B_2C_2, C_2A_1 of this triangle lie respectively the points C_3, A_2, B_3 forming another triangle inscribed in the given triangle. On the sides C_3A_2, A_2B_3, B_3C_3 of the second triangle lie the points B_1, C_1, A_3 which form another triangle inscribed in the second. The triangle inscribed in this third triangle is the triangle $A_1B_2C_2$, with which we started.

From the above consideration we are led to enunciate the following theorem :

If a triangle Δ is inscribed in a triangle Δ', then there are an unlimited number of triangles which are simultaneously inscribed in Δ' and circumscribed to Δ.

The proof of this theorem depends on the fact that it is possible to make an unlimited number of constructions of Pappus configuration with the six given vertices. For, let Δ and Δ' be $A_1B_2C_2$ and $C_3A_2B_3$ respectively. Through A_1 draw *any* line s and let A_2C_3, A_2B_3 meet s in B_1, C_1 respectively. If now C_1B_2 and B_1C_2 intersect in A_3, then, by Pappus theorem, A_3 is a point on B_3C_3. Hence $B_1C_1A_3$ is *one* of the required triangles.

37. Correlation and Polarity. Consider a transformation of the form

$$\begin{aligned} \rho u_1 &= a_1x_1+a_2x_2+a_3x_3 \\ \rho u_2 &= b_1x_1+b_2x_2+b_3x_3 \qquad | abc | \neq 0, \qquad (10.7) \\ \rho u_3 &= c_1x_1+c_2x_2+c_3x_3 \end{aligned}$$

transforming a point (x_1, x_2, x_3) into a line (u_1, u_2, u_3). Since $| abc | \neq 0$, (10.7) has its inverse which transforms lines into points. By (10.7), a point (p_i) is transformed into the line whose coordinates are $(\sum a_jp_j, \sum b_jp_j, \sum c_jp_j)$. Denote these line coordinates by (p_i'); similarly, denote the coordinates of the line, into which a point (q_i) is transformed, by (q_i'). Then, as in § 34.1, the point $(\mu p_i+\nu q_i)$ is transformed into the line whose coordinates are

$$\Big(\sum_j a_j(\mu p_j+\nu q_j),\ \sum_j b_j(\mu p_j+\nu q_j),\ \sum_j c_j(\mu p_j+\nu q_j)\Big),$$

that is, into the line whose coordinates are $(\mu p_i'+\nu q_i')$. This shows that collinear points are transformed into concurrent lines, and that the cross-ratio of four points is equal to the cross-ratio of the four corresponding lines into which the four points are transformed, and vice versa.

A transformation of the form (10.7) is called a *correlation*. Thus, a correlation transforms points into lines, hence lines into points, and preserves cross-ratio. The inverse of a correlation is also a correlation. If a correlation $(x_1, x_2, x_3) \to (u_1, u_2, u_3)$ is followed by a correlation $(u_1, u_2, u_3) \to (x'_1, x'_2, x'_3)$, the product is a collineation $(x_1, x_2, x_3) \to (x'_1, x'_2, x'_3)$.

Take a point (ξ_i) on the line (u_i). Then, $\sum \xi_iu_i = 0$ and hence, by (10.7), $\sum x_iv_i = 0$, where

$$\begin{aligned} \sigma v_1 &= a_1\xi_1+b_1\xi_2+c_1\xi_3 \\ \sigma v_2 &= a_2\xi_1+b_2\xi_2+c_2\xi_3 \qquad (10.8) \\ \sigma v_3 &= a_3\xi_1+b_3\xi_2+c_3\xi_3 \end{aligned}$$

the v's being thus dependent on the ξ's. So, for different points (ξ_i) on the line (u_i), we obtain different lines (v_i) through the point (x_i). The transformation (10.7) and (10.8) are, in general, different as the two matrices of the coefficients in the two transformations are, in general, different. The transformation (10.7) would transform (ξ_i) into a line (w_i) which would be generally different from (v_i). Now, suppose that the two transformations (10.7) and (10.8) are the same; that is, suppose that the two lines (v_i) and (w_i) coincide. The condition for this is that the two matrices of the coefficients should differ only by a common factor. So we may put

$$a_2 = b_1, \quad a_3 = c_1, \quad b_3 = c_2. \tag{10.9}$$

When this is satisfied, the transformation (10.7) is called a *polarity*. In a polarity, the corresponding point and line, *i.e.*, a point and the line into which the point is transformed, are called the *pole* and the *polar* respectively. And we have just seen that if a point (ξ_i) lies on a line (u_i), then the polar (v_i) of (ξ_i) passes through the pole (x_i) of (u_i). The points (ξ_i) and (x_i) are called *conjugate points* and the lines (u_i), (v_i) are called *conjugate lines*. These statements agree with what has been said in § 13. When (10.9) is satisfied, (10.7) gives a polarity even if $|\,abc\,| = 0$. A polarity may accordingly be written as

$$\begin{aligned} \rho u_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \rho u_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \qquad a_{ij} = a_{ji}, \\ \rho u_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \tag{10.10}$$

or, in the compact form,

$$\rho u_i = \sum_j a_{ij}x_j, \quad i = 1, 2, 3$$

The polarity is said to be *degenerate or nondegenerate* according as the determinant $|\,a_{ij}\,|$ of the coefficients does or does not vanish. Suppose that the polarity is nondegenerate, so that $|\,a_{ij}\,| \neq 0$. Then the inverse of (10.10) is the polarity (cf. the inverses in § 34)

$$\sigma x_i = \sum_{j=1}^{3} A_{ji}\,u_j, \quad i = 1, 2, 3, \quad A_{ij} = A_{ji},$$

where A_{ij} are the cofactors of a_{ij} in the determinant $|\,a_{ij}\,|$. It can be seen that the resultant of two polarities transforming (x_1, x_2, x_3) into (u_1, u_2, u_3) and then (u_1, u_2, u_3) into (x_1', x_2', x_3') is a collineation. In a polarity the cross-ratio is preserved.

If a triangle be such that each side is the polar of the opposite vertex, hen the triangle is called a *polar* (or *self-polar* or *self-conjugate*) *triangle*. In constructing a polar triangle, we take any point P and its polar p; on p take any point Q; then q, the polar of Q, must pass through P; let

p and q intersect in R ; then r, the polar of R, must be the line PQ. Thus PQR is a polar triangle.

Let us now suppose that a point (x_i) lies on its own polar (u_i). So, $\sum u_ix_i = 0$. Therefore, from (10.10), we have

$$\sum_{i,j=1}^{3} a_{ij}x_ix_j = 0, \quad a_{ij} = a_{ji}$$

This equation is the same as (9.15). Hence the locus of (x_i) is a curve of the second degree or a conic. This conic is called the *nucleus* of the polarity (10.10). The nucleus is a conic locus or a conic envelope according as we consider the conic as the locus of the points conjoint with their polars or the envelope of lines conjoint with their poles.

Conversely, suppose that we start with a nondegenerate conic $\sum a_{ij}x_ix_j = 0$, $a_{ij} = a_{ji}$ and let $P = (\xi_i)$ be any point. The polar of P with respect to the conic is, by (9.18),

$$\sum_{i,j} a_{ij}\,\xi_ix_j = 0$$

or

$$\Big(\sum_j a_{1j}\xi_j\Big)x_1 + \Big(\sum_j a_{2j}\xi_j\Big)x_2 + \Big(\sum_j a_{3j}\xi_j\Big)x_3 = 0$$

or

$$v_1x_1 + v_2x_2 + v_3x_3 = 0, \text{ say,}$$

where

$$\rho v_i = \sum_j a_{ij}\xi_j, \quad a_{ij} = a_{ji}, \quad i = 1, 2, 3.$$

This is the polarity (10.10). Thus, the conic gives rise to a polarity. We can therefore speak of pole and polar with respect to a polarity transformation or with respect to a conic and we say that a polarity generates a *polar field* in the sense that to each point there corresponds a polar and to each line there corresponds a pole. We shall take up the discussion of polar field in a subsequent chapter.

Consider two triangles ABC, $A'B'C'$ such that polars of the vertices A, B, C of one triangle are the sides $B'C'$, $C'A'$, $A'B'$ respectively of the other. It follows that the polars of A', B', C' are BC, CA, AB respectively. Two such triangles are called *relative polar* triangles. Referring to Desargues' triangles of the last article, we have the following theorem :

Two relative polar triangles are perspective.

Let (10.10) be the polarity and let the coordinate system be so chosen (by collineation) that the coordinates of the vertices A, B, C

of one triangle are (1, 0, 0), (0, 1, 0), (0, 0, 1) respectively. Then the coordinates of the sides $B'C'$, $C'A'$, $A'B'$ of the other triangle are respectively

$$(a_{11}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{13}, a_{23}, a_{33})$$

Therefore the coordinates of the vertices A', B', C' are respectively

$$(A_{11}, A_{21}, A_{31}), (A_{12}, A_{22}, A_{32}), (A_{13}, A_{23}, A_{33})$$

where, as above, A_{ij} are the cofactors of a_{ij} in $| a_{ij} |$. Hence, the coordinates of the lines AA', BB', CC' are respectively

$$(0, A_{31}, -A_{21}), (-A_{32}, 0, A_{12}), (A_{23}, -A_{13}, 0)$$

Accordingly, since the determinant of the last set of coordinates vanishes, *i.e.*,

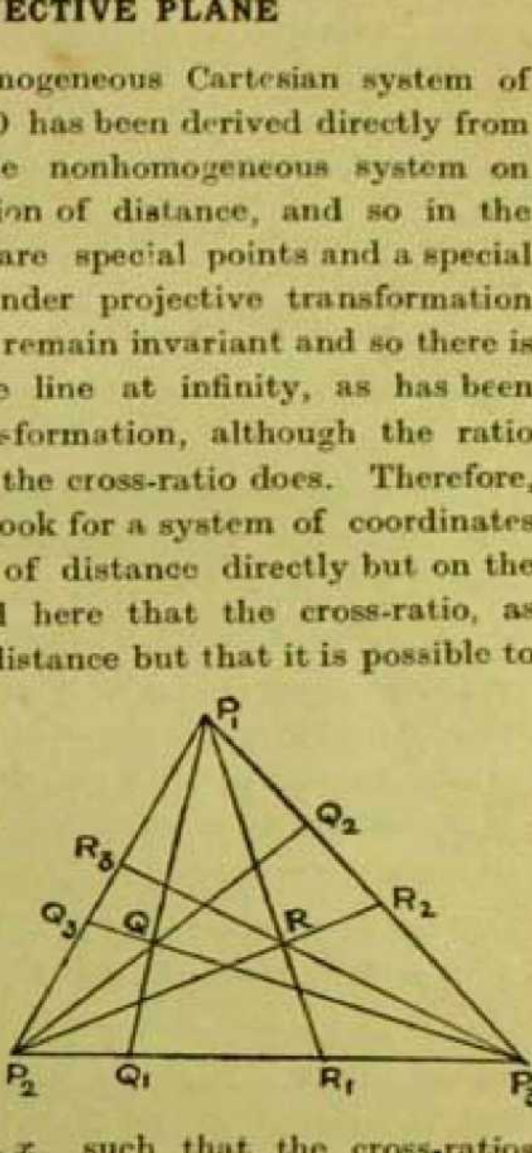
$$\begin{vmatrix} 0 & A_{31} & -A_{21} \\ -A_{32} & 0 & A_{12} \\ A_{23} & -A_{13} & 0 \end{vmatrix} = A_{12}A_{13}A_{23} \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 0,$$

the three lines AA', BB', CC' are concurrent and so the three intersections of pairs of corresponding sides are collinear.

CHAPTER XI

GEOMETRY IN THE PROJECTIVE PLANE

38. Projective coordinates. The homogeneous Cartesian system of coordinates (x_1, x_2, x_3) introduced in § 30 has been derived directly from the nonhomogeneous system (x, y). The nonhomogeneous system on the other hand, depends directly on the notion of distance, and so in the homogeneous system thus derived there are special points and a special line. Now, we have seen in § 34.1 that under projective transformation (*i.e.*, collineation), the distance does not remain invariant and so there is nothing special about the points and the line at infinity, as has been noticed in §§ 32, 36. But, under this transformation, although the ratio of distances does not remain invariant, the cross-ratio does. Therefore, in the projective geometry, it is natural to look for a system of coordinates which would depend not on the notion of distance directly but on the notion of cross-ratio. It may be remarked here that the cross-ratio, as defined in § 6, depends, on the notion of distance but that it is possible to avoid this notion by introducing the cross-ratio in a different manner. However we do not propose to do so here.



Let P_1, P_2, P_3 be the vertices of a triangle and Q a point not lying on any side of the triangle. Let P_1Q, P_2Q, P_3Q meet the opposite sides in the points Q_1, Q_2, Q_3 respectively. Take any point R, and let P_1R, P_2R, P_3R meet the opposite sides in the points R_1, R_2, R_3 respectively. We now introduce three real numbers x_1, x_2, x_3 such that the cross-ratios have the values

$$(P_2P_3, Q_1R_1) = x_2/x_3,\ (P_3P_1, Q_2R_2) = x_3/x_1,\ (P_1P_2, Q_3R_3) = x_1/x_2 \quad (11.1)$$

The ordered triad of numbers (x_1, x_2, x_3) are called the *projective coordinates of the point R* with reference to the traingle $P_1P_2P_3$ and the point Q. It is evident that the projective coordinates are homogeneous coordinates and it is to be noticed that the product of the cross-ratios given above is equal to unity.

In justifying the definition (11.1) of the coordinates of a point, let us show that given the triangle $P_1P_2P_3$ and the point Q, we can always find three numbers x_1, x_2, x_3 for every point R of the plane, and conversely when the three numbers are given, R is determined uniquely.

Firstly, suppose that R does not lie on any side of the triangle $P_1P_2P_3$. Then each of the cross-ratios $(P_2P_3,\ Q_1R_1)$, $(P_3P_1,\ Q_2R_2)$, $(P_1P_2,\ Q_3R_3)$ is defined and the product of them is equal to unity. Consequently, we can solve the three equations (11.1) for the unknowns x_1, x_2, x_3, none of which can be zero; the general solution is given by $(\rho x_1, \rho x_2, \rho x_3)$, $\rho \neq 0$. Conversely, if three numbers x_1, x_2, x_3, none of which is zero, are given, then the cross-ratios are known; that is, the points R_1, R_2, R_3 are known. And, since the product of the cross-ratios is equal to unity, the lines P_1R_1, P_2R_2, P_3R_3 are concurrent (Theo. I § 36) in the required point R not lying on any side of the triangle. In particular, if R coincides with Q, each of the cross-ratios is equal to unity and the coordinates of Q are therefore (1, 1, 1).

Secondly, suppose that R lies on one side of the triangle, say on P_2P_3, not coinciding with any vertex. Then R_1 coincides with R, R_2 with P_3 and R_3 with P_2. So,

$$x_2/x_3 = (P_2P_3,\ Q_1R), \qquad x_3/x_1 = 1/0, \qquad x_1/x_2 = 0/1$$

Therefore, $x_1 = 0, x_2 \neq 0, x_3 \neq 0$ where x_2/x_3 is defined by the cross-ratio. Thus R has the coordinates $(0, x_2, x_3)$, where $x_2x_3 \neq 0$. Conversely, given three numbers $(0, x_2, x_3)$, $x_2x_3 \neq 0$, the point R is determined on the side P_2P_3. Similarly, if R is a point of P_3P_1 or of P_1P_2, other than a vertex, its coordinates are $(x_1, 0, x_3)$, $x_1x_3 \neq 0$ or $(x_1, x_2, 0)$, $x_1x_2 \neq 0$, respectively.

Lastly, suppose that R coincides with a vertex, say with P_1. Then R_1 is undetermined, R_2 and R_3 coincide with P_1. So,

$$x_2/x_3 \text{ is undefined}, \qquad x_3/x_1 = 0/1, \quad x_1/x_2 = 1/0.$$

These equations are satisfied if we take $x_1 = 1$, $x_2 = 0$, $x_3 = 0$. Thus the coordinates of P_1 are (1, 0, 0). Conversely, given three numbers $(\rho, 0, 0)$, $\rho \neq 0$, the point R coincides with P_1. Similarly, the coordinates of P_2, P_3 are (0, 1, 0), (0, 0, 1) respectively.

Thus the justification for the definition of projective point coordinates is established. It follows that the equations of the lines P_2P_3, P_3P_1, P_1P_2 are $x_1 = 0, x_2 = 0, x_3 = 0$ respectively.

We introduce *projective line coordinates* in exactly the same way as has been done in § 31. If (x_1, x_2, x_3), (u_1, u_2, u_3) are the projective coordinates of a point and a line respectively, the condition that the point and the line are conjoint is

$$u_1x_1 + u_2x_2 + u_3x_3 = 0.$$

So the coordinates of the lines P_2P_3, P_3P_1, P_1P_2 are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively, and the equations in line coordinates of the points P_1, P_2, P_3 are $u_1 = 0$, $u_2 = 0$, $u_3 = 0$ respectively. The points $(0, 1, -1)$, $(-1, 0, 1)$, $(1, -1, 0)$ on the three sides P_2P_3, P_3P_1, P_1P_2 respectively of the triangle lie on a line q (say) whose equation in point coordinates is

$$x_1 + x_2 + x_3 = 0.$$

So, the coordinates of the line q are $(1, 1, 1)$.

The triangle $P_1P_2P_3$ is called the *triangle of reference* or *the fundamental triangle*; the point Q with coordinates $(1, 1, 1)$ is spoken of as the *unit point* and the line q with coordinates $(1, 1, 1)$ the *unit line*.

It follows from above that *given four points forming a quadrangle, we can introduce a system of projective coordinates by taking the triangle formed by three of the points as the triangle of reference and the remaining point as the unit point.* We may build the theory of projective line coordinates independently by dualising the theory of projective point coordinates as follows :

Let p_1, p_2, p_3 be the three sides of a triangle and q a line not passing through any vertex of the triangle. Let q_1, q_2, q_3 be the lines joining the points p_1q, p_2q, p_3q with the opposite vertices of the triangle. Take any other line r and let r_1, r_2, r_3 be the lines joining the points p_1r, p_2r, p_3r with the opposite vertices. If now we take three numbers u_1, u_2, u_3 such that

$$(p_2p_3, q_1r_1) = u_2/u_3, \quad (p_3p_1, q_2r_2) = u_3/u_1, \quad (p_1p_2, q_3r_3) = u_1/u_2,$$

then (u_1, u_2, u_3) are called the projective coordinates of the line r with reference to the triangle $p_1p_2p_3$ and the line q. It follows from the definition that the projective coordinates of the lines p_1, p_2, p_3, q are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ respectively.

Thus, *given four lines forming a quadrilateral, we can introduce a system of projective coordinates by taking the triangle formed by three of the lines as the triangle of reference and the remaining line as the unit line.*

In view of the relation (10.5) in § 36, it may be seen that *the homogeneous Cartesian coordinates are special projective coordinates*, one side of the fundamental triangle being the line at infinity. If then, we make no distinction between an ordinary line and the line at infinity, between an ordinary point and a point at infinity, all results that have been obtained in §§ 33, 34 in homogeneous Cartesian coordinates hold also in projective coordinates.

The projective coordinates of the points of a row are obtained by taking three points P_∞, P_0, Q as the fundamental points of the row. If P is a point of the row and the cross-ratio

$$(P_\infty P_0, Q P) = x_1/x_2,$$

then (x_1, x_2) are the projective coordinates of P. In particular, the projective coordinates of P_∞, P_0, Q are (1, 0), (0, 1), (1, 1) respectively.

Similarly, the projective coordinates of the lines of a pencil are obtained by taking three lines p_∞, p_0, q as the fundamental lines of the pencil. The projective coordinates (u_1, u_2) of a line p of the pencil are then given by

$$(p_\infty p_0, qp) = u_1/u_2.$$

The projective coordinates of the points of a row or of the lines of a pencil belong to one-dimensional projective geometry (§ 35).

38.1. Transformation of projective coordinates. Collineation. Let us take a triangle of reference $P_1P_2P_3$ and a unit point Q, and let the projective coordinates of a point R with reference to the triangle and the unit point be (x_i'). Also, let the homogeneous Cartesian coordinates of P_1, P_2, P_3, Q and R be (a_i), (b_i), (c_i), (d_i) and (x_i) respectively. The coordinates of the lines P_1Q and P_2P_3 are then

$$(a_2d_3 - d_2a_3,\ a_3d_1 - d_3a_1,\ a_1d_2 - d_1a_2) \text{ and } (b_2c_3 - c_2b_3,\ b_3c_1 - c_3b_1,\ b_1c_2 - c_1b_2)$$

respectively. Therefore, the coordinates of Q_1, the point of intersection of these two lines, are

$$(\mu b_1 + \nu c_1, \quad \mu b_2 + \nu c_2, \quad \mu b_3 + \nu c_3),$$

where

$$\mu = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \equiv |c\,a\,d|, \quad \nu = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \equiv |abd|.$$

Similarly, the coordinates of R_1, the point of intersection of the lines P_1R and P_2P_3, are

$$(\mu' b_1 + \nu' c_1, \quad \mu' b_2 + \nu' c_2, \quad \mu' b_3 + \nu' c_3),$$

where, in accordance with the above notation,

$$\mu' = |\,c\,a\,x\,|, \quad \nu' = |\,a\,b\,x\,|.$$

Hence, the cross-ratio

$$(P_2P_3, Q_1R_1) = \frac{\nu\mu'}{\mu\nu'} = \frac{|\,a\,b\,d\,|\ |\,c\,a\,x\,|}{|\,c\,a\,d\,|\ |\,a\,b\,x\,|} = \frac{|\,d\,a\,b\,|\ |\,x\,c\,a\,|}{|\,d\,c\,a\,|\ |\,x\,a\,b\,|}$$

Or, by (11.1),
$$\frac{x_2'}{x_3'}=\frac{|x\,c\,a|}{|d\,c\,a|}\ :\ \frac{|x\,a\,b|}{|d\,a\,b|}.$$

Similarly
$$\frac{x_3'}{x_1'}=\frac{|x\,a\,b|}{|d\,a\,b|}\ :\ \frac{|x\,b\,c|}{|d\,b\,c|},\quad \frac{x_1'}{x_2'}=\frac{|x\,b\,c|}{|d\,b\,c|}\ :\ \frac{|x\,c\,a|}{|d\,c\,a|}.$$

Hence
$$\rho x_1'=\frac{|x\,b\,c|}{|d\,b\,c|},\qquad \rho x_2'=\frac{|x\,c\,a|}{|d\,c\,a|},\qquad \rho x_3'=\frac{|x\,a\,b|}{|d\,a\,b|}. \tag{11.2}$$

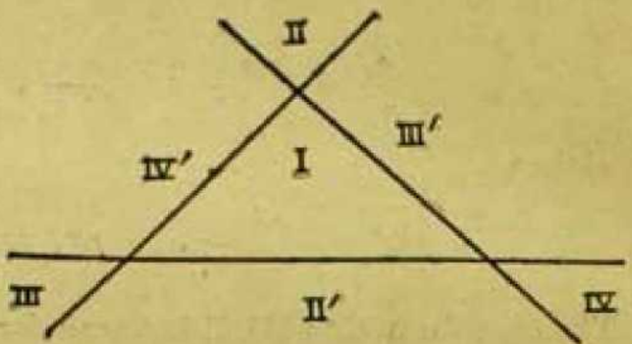
The transformation (11.2) is a transformation from homogeneous Cartesian to projective coordinates. Since the a's, b's, c's, d's are constants, the numerators are linear homogeneous functions of the x's and the denominators are constants, different from zero. Therefore (11.2) is a linear homogeneous transformation of the form (10.1) and is thus a collineation. It follows that the transformation from homogeneous Cartesian to projective is a collineation, and so is its inverse. Hence, *the transformation from one system of projective coordinates to another is a collineation.*

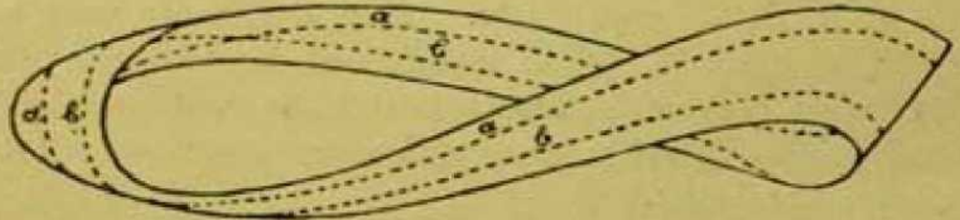
Let the points P_1, P_2, P_3, Q be transformed by a collineation into the points A, B, C, D respectively. Then any point with projective coordinates $(x_1, x_2, x_3) = (a, b, c)$ is transformed into a point which has the coordinates (a, b, c) with respect to ABC as the triangle of reference and D as unit point. Therefore, *every collineation represents a transformation of projective coordinates.*

Hence, a collineation of the projective plane is of the same form as (10.1), and can be written as
$$\rho x_i' = \sum a_{ij}\, x_j,\quad i = 1, 2, 3,\quad |\, a_{ij}\,| \neq 0 \tag{11.3}$$
in point coordinates. In line coordinates, (11.3) is
$$\sigma u_i' = \sum A_{ij}\, u_j,\quad i = 1, 2, 3,\quad |\, A_{ij}\,| \neq 0 \tag{11.3'}$$
where A_{ij} are the cofactors of a_{ij} in $|\, a_{ij}\,|$. The inverses are respectively
$$\rho' x_i = \sum A_{ji}\, x'_j,\quad \sigma' u_i = \sum a_{ji}\, u'_j. \tag{11.3''}$$
Sometimes it is useful to write these equations in the matrix form as in next article.

We have seen in § 28 that a line in the extended Cartesian plane is to be regarded as closed ; this is also true in the projective plane, where, as we have said in § 32, we make no distinction between an ordinary point and a point at infinity. So, the nature of the plane, whether extended or projective, remains the same.

In order to understand the nature of the projective plane, consider a figure formed by three nonconcurrent lines. If the plane were just the ordinary (Euclidean) plane, a triangle would divide the plane into seven regions, I, II, II′, III, III′, IV, IV′ as indicated in the figure. But in the projective plane, since a line is closed, the regions II and II together make up a triangle and so constitute one region. So do III and III′, also IV and IV′. Thus the three nonconcurrent lines divide the plane into four regions only. Further, since two lines a, b in a projective plane always intersect in only one point, the intuitive conception of a projective plane may be assisted by a model of one-sided surface due to Möbius. The model is constructed by cutting out a rectangular strip of paper, giving it a half-twist and pasting together the two ends, as is shown in the diagram below :





39. Classification of polarities and conics in the projective plane. Let us start with a point-to-line correspondence defined by the equations of the form (10.10), namely

$$\rho u_i = \sum c_{ik} x_k, \quad i = 1, 2, 3, \quad c_{ik} = c_{ki}. \tag{11.4}$$

These equations may be written in the matrix form as

$$(u_i) = (c_{ik})(x_k).$$

Applying an arbitrary collineation of the form (11.3), namely

$$(x'_i) = (a_{ik})(x_k), \quad (u'_i) = (A_{ik})(u_k)$$

and using the inverse, we obtain

$$\sigma(a_{ji})(u'_j) = (c_{ik})(A_{\nu k})(x'_\nu).$$

Hence

$$\sigma(A_{\mu i})(a_{ji})(u'_j) = (A_{\mu i})(c_{ik})(A_{\nu k})(x'_\nu)$$

or

$$\varpi(u'_\mu) = (c'_{\mu\nu})(x'_\nu), \quad \text{where} \quad c'_{\mu\nu} = \sum_{i,k} A_{\mu i}\, c_{ik} A_{\nu k} \tag{11.4'}$$

and ϖ is an arbitrary constant. But

$$c'_{\nu\mu} = \sum A_{\nu i} c_{ik} A_{\mu k} = \sum A_{\nu k} c_{ki} A_{\mu i} = c'_{\mu\nu}$$

and the rank of (c_{ik}) = the rank of (c'_{ik}), since $|A_{ik}| \neq 0$. Thus we state

The correspondence (11.4) is transformed by an arbitrary collineation into a correspondence of the same kind, the symmetric matrix (c_{ik}) *being replaced by a symmetric matrix* (c'_{ik}). *Further, the rank of the matrix of the correspondence is not altered by collineation.*

Corresponding to the notions of § 37, we shall call the correspondence (11.4) a *polarity of the projective plane*; the point (x_1, x_2, x_3) and the line (u_1, u_2, u_3) are the pole and the polar.

Let the point (ξ_1, ξ_2, ξ_3) be situated on the polar of (x_1, x_2, x_3). Then

$$0 = \sum \xi_i u_i = \sum c_{ik}\, \xi_i x_k = \sum c_{ki}\, \xi_i x_k$$

From the symmetry of this equation, it follows that if a point Q is situated on the polar of a point P and if Q has a polar, then P is situated on the polar of Q. It may happen that Q has no polar when its coordinates substituted in (11.4) make $u_1 = u_2 = u_3 = 0$. This cannot occur indeed if $|c_{ik}| \neq 0$. As in § 37, the condition that the point (x_i) is situated on its own polar is

$$\sum c_{ik}\, x_i x_k = 0 \tag{11.5}$$

This condition is both necessary and sufficient for those points which have polars. The equation (11.5) is a homogeneous equation of the second degree and every such equation can be written in this form; the coefficients $c_{ik} = c_{ki}$ are given by it uniquely (there being only a common arbitrary factor). Thus, the polarity (11.4) and its nucleus (11.5), *i.e.*, the conic generated by the polarity, determine one another uniquely. Every invariant of one of them is an invariant of the other. Hence, *the rank of the matrix* (c_{ik}) *is an invariant of the conic (11.5) for every collineation.*

The above consideration leads us to classify the polarities and the conics simultaneously from the projective point of view as follows:

(1) *Rank of* $(c_{ik}) = 0$. In this case every coefficient $c_{ik} = 0$; so, no point has a polar and every point of the plane satisfies (11.5). This is a trivial case.

(2) *Rank of* $(c_{ik}) = 1$. In this case the homogeneous equations $\sum c_{ik} x_k = 0$, $i = 1, 2, 3$, have two independent solutions other than $(0, 0, 0)$. All the solutions therefore form a row of points, these points having no polar. As in the derivation of (11.4′), we now apply a collineation transforming the given polarity into another such that the base of

this row of points is the line $x'_3 = 0$, *i.e.*, the transformed polarity is such that for every point $(x'_1, x'_2, 0)$ we shall have $u'_1 = u'_2 = u'_3 = 0$. Then the coefficients of the first and the second rows of the transformed matrix (c'_{ik}) shall be zero, as $c'_{13} = c'_{31}$, $c'_{23} = c'_{32}$. Therefore, only c'_{33} shall be different from zero ; and, without loss of generality, we may put $c'_{33} = 1$. Dropping the dashes, the transformed polarity takes the *normal* form

$$u_1 = 0, \;\; u_2 = 0, \;\; \rho u_3 = x_3$$

That the polarity can in this case be so transformed depends on the fact that the rank of the matrix remains unaltered.

Hence, in this case there exists one line such that the points of this line have no polar, whereas every other point has this line as its polar. The equation of the nucleus conic is therefore transformed into the *normal* form

$$x_3^2 = 0$$

(3) *Rank of* $(c_{ik}) = 2$. In this case, the equations $\sum c_{ik}x_k = 0$, $i = 1, 2, 3$, have only one solution other than $(0, 0, 0)$; that is, there exists only one point which has no polar. As before, we apply a suitable collineation so that this point is the point $(0, 0, 1)$. So, the polarity can be written as

$$\begin{aligned} \rho u_1 &= c_{11}x_1 + c_{12}x_2 \\ \rho u_2 &= c_{12}x_1 + c_{22}x_2 \qquad\qquad c_{11}c_{22} - c_{12}^2 \neq 0. \\ \rho u_3 &= 0 \end{aligned}$$

Hence, every polar passes through the point $(0, 0, 1)$ and the points of every line passing through $(0, 0, 1)$ have the same polar. Thus the polarity can be regarded as a correspondence between the lines of the pencil with centre $(0, 0, 1)$. The nucleus (11.5) is, in this case,

$$c_{11}x_1^2 + 2c_{12}x_1x_2 + c_{22}x_2^2 = 0, \quad c_{11}c_{22} - c_{12}^2 \neq 0$$

By a suitable transformation, not altering x_3, we can transform this equation into either of the *normal* forms

$$x_1^2 + x_2^2 = 0 \quad \text{or} \quad x_1^2 - x_2^2 = 0$$

corresponding to the sign of $c_{11}c_{22} - c_{12}^2$ which cannot be altered. In the first case the nucleus has no real branch and the polarity is represented by the *normal* form

$$\rho u_1 = x_1, \;\; \rho u_2 = x_2, \;\; \rho u_3 = 0.$$

This polarity generates an elliptic involution in the lines of the pencil with centre $(0, 0, 1)$. In the second case the nucleus consists of the two lines

$$x_1 - x_2 = 0, \;\; x_1 + x_2 = 0$$

intersecting in the point (0, 0, 1) and the polarity is represented by the *normal* form

$$\rho u_1 = x_1, \quad \rho u_2 = -x_2, \quad \rho u_3 = 0$$

This polarity generates hyperbolic involution in the lines of the pencil with centre (0, 0, 1) ; the double lines are the two lines of the nucleus. [See the two possibilities under (8.5)]

(4) *Rank of* $(c_{ik}) = 3$. In this case there is no solution of the equations $\sum c_{ik}x_k = 0$, $i = 1, 2, 3$, other than (0, 0, 0) ; so, every point has a polar. We introduce a triangle which is self-polar with respect to this polarity [there is no such triangle in the cases (1), (2), (3) above]. Such a triangle exists, because the nucleus-conic is nondegenerate. If a self-polar triangle is chosen as the triangle of reference, *i.e.*, with vertices (0, 0, 1), (0, 1, 0), (1, 0, 0), the equation of the nucleus is reduced to the form

$$c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 = 0$$

Put $$c_{11} = \pm a^2, \quad c_{22} = \pm b^2, \quad c_{33} = \pm c^2,$$

and, without loss of generality, assume that at least two of the signs are positive and that they are the signs prefixed to a^2 and b^2. Then putting

$$ax_1 = x'_1, \quad bx_2 = x'_2, \quad cx_3 = x'_3$$

and dropping the dashes, we get the equations of the nuclei in the *normal* forms

$$x_1^2 + x_2^2 + x_3^2 = 0$$

and $$x_1^2 + x_2^2 - x_3^2 = 0$$

The corresponding polarities are

$$\rho u_1 = x_1, \quad \rho u_2 = x_2, \quad \rho u_3 = x_3$$

and $$\rho u_1 = x_1, \quad \rho u_2 = x_2, \quad \rho u_3 = -x_3$$

In the first case however the nucleus is without real trace. All cases have been considered. There exist therefore the following *classes* of polarities and their corresponding conics (nuclei) :

Cases	Ranks of matrices	Polarities	Conics
1	1	$u_1 = u_2 = 0,\ \rho u_3 = x_3$	$x_3^2 = 0$ (11.6)
2	2	$\rho u_1 = x_1,\ \rho u_2 = x_2,\ u_3 = 0$	$x_1^2 + x_2^2 = 0$ (11.7)
3	2	$\rho u_1 = x_1,\ \rho u_2 = -x_2,\ u_3 = 0$	$x_1^2 - x_2^2 = 0$ (11.8)
4	3	$\rho u_1 = x_1,\ \rho u_2 = x_2,\ \rho u_3 = x_3$	$x_1^2 + x_2^2 + x_3^2 = 0$ (11.9)
5	3	$\rho u_1 = x_1,\ \rho u_2 = x_2,\ \rho u_3 = -x_3$	$x_1^2 + x_2^2 - x_3^2 = 0$ (11.10)

From (11.10) it follows that all nondegenerate (real) conics are *equivalent* in the projective geometry in the sense that one such conic can be transformed into any other by a suitable collineation. We may express this by saying that *all circles, ellipses, hyperbolas and parabolas are projective to each other.* Therefore, we do not discriminate between these curves.

The equation of a nondegenerate (real) conic can be put into another form. Let

$$\sum a_{ij}x_ix_j = 0, \quad a_{ij} = a_{ji},$$

be the equation of a conic in projective coordinates. Take three points A_1, A_2, A_3 on the conic forming a triangle. By a collineation, transform the coordinates (x_1, x_2, x_3) into (x'_1, x'_2, x'_3) so that $A_1\ A_2\ A_3$ becomes the triangle of reference; so, the equation of the conic takes the form

$$\sum a'_{ij}x'_ix'_j = 0, \quad a'_{ij} = a'_{ji}.$$

As the conic passes through the vertices of the triangle of reference whose coordinates are (1, 0, 0), (0, 1, 0), (0, 0, 1), we must have

$$a'_{11} = a'_{22} = a'_{33} = 0$$

So, the equation of the conic takes the form

$$a'_{12}x'_1x'_2 + a'_{13}x'_1x'_3 + a'_{23}x'_2x'_3 = 0$$

None of the coefficients is equal to zero as the conic is supposed to be nondegenerate. We can therefore apply the collineation

$$\rho x''_1 = \frac{1}{a'_{23}}x'_1, \quad \rho x''_2 = \frac{1}{a'_{13}}x'_2, \quad \rho x''_3 = \frac{1}{a'_{12}}x'_3$$

The equation of the conic reduces to the form (dropping the dashes)

$$x_1x_2 + x_1x_3 + x_2x_3 = 0. \qquad (11.10')$$

The equation (11.10′) is the required alternative form of (11.10).

Dual polarities. Let us now consider a line-to-point correspondence and discuss polarity and nuclei arising therefrom. For this purpose we have only to interchange the $x-$ and the $u-$ coordinates in our discussion of the point-to-line correspondence (11.4) given before. When we do so, we obtain, as above, five normal forms of polarities and their nuclei; the nuclei are here envelopes of the second class. It must be noticed that in the first three of the five cases, the interpretation of the polarities are different in the two dual cases. E.g., in the first case (11.6), where the rank of the matrix is one, there are an infinity of points $(x_3 \neq 0)$ which have the same polar and the corresponding nucleus-conic is a line (repeated twice); but in the corresponding dual case only one point (0, 0, 1) is a pole and the nucleus-envelope is the pencil which has this point as the centre. However, in the non-trivial cases, the dual polarities furnish the same correspondence between points and lines. The nucleus-envelope consists therefore of the polars which pass through their poles. Thus, in the cases (11.9) and (11.10), where the rank of the matrix is three, the normal forms of the envelopes are $u_1^2+u_2^2+u_3^2=0$ which contains no real line and $u_1^2+u_2^2-u_3^2=0$ which consists of the tangents to $x_1^2+x_2^2-x_3^2=0$.

When the equations are not given in the normal forms we have the following results :

Let C_{ik} be the cofactors of $c_{ik}=c_{ki}$ in $|\,c_{ik}\,|$ and let the rank of (c_{ik}) be three. Then

Polarities	Nuclei
$\rho u_i = \sum c_{ik}x_k$	$\sum c_{ik}x_ix_k=0$
$\sigma x_i = \sum C_{ik}u_k$	$\sum C_{ik}u_iu_k=0.$

40. Quadratic dependence of points. Consider a conic given by the general equation $\sum a_{ij}x_ix_j=0$. The six products

$$x_1x_1,\quad x_1x_2,\quad x_2x_2,\quad x_1x_3,\quad x_2x_3,\quad x_3x_3$$

depend on the three quantities x_1, x_2, x_3 and so they are not arbitrary.

Let Q_ν with coordinates $(c_{\nu1}, c_{\nu2}, c_{\nu3})$ be six points for $\nu=1, 2, 3, 4, 5, 6$ ($c_{\nu i}$ is not necessarily equal to $c_{i\nu}$) ; and let $|\,c_{\nu i}\,c_{\nu j}\,|$ stand for the determinant

$$\begin{vmatrix} c_{11}c_{11} & c_{11}c_{12} & c_{12}c_{12} & c_{11}c_{13} & c_{12}c_{13} & c_{13}c_{13} \\ c_{21}c_{21} & c_{21}c_{22} & c_{22}c_{22} & c_{21}c_{23} & c_{22}c_{23} & c_{23}c_{23} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{61}c_{61} & c_{61}c_{62} & c_{62}c_{62} & c_{61}c_{63} & c_{62}c_{63} & c_{63}c_{63} \end{vmatrix}$$

Then, if the six points Q_ν lie on the conic, we shall have the six equations

$$\sum_{i,j=1}^{3} a_{ij}c_{\nu i}c_{\nu j}=0, \quad \nu=1, 2, \ldots., 6$$

satisfied simultaneously. Since a_{ij} cannot all be zero, the condition that the six points lie on the conic is

$$|\,c_{\nu i}\,c_{\nu j}\,|=0$$

On the other hand, suppore that Q_6 is a variable point (x_1, x_2, x_3) ; then developing the determinant $|\,c_{\nu i}\,c_{\nu j}\,|$ in terms of the elements of the sixth row, we have

$$C_{11}\,x_1x_1+C_{12}\,x_1x_2+C_{22}\,x_2x_2+C_{13}\,x_1x_3+C_{23}\,x_2x_3+C_{33}\,x_3x_3=0,$$

where C_{ij} are the cofactors of $c_{si}\, c_{sj}$ in the determinant $|c_{si}\, c_{sj}|$. This is an equation of the second degree unless each C_{ij} is zero. Hence, the five points Q_1, Q_2, Q_3, Q_4, Q_5 determine a conic uniquely if the rank of the following matrix is equal to five

$$\begin{pmatrix} c_{11}c_{11} & c_{11}c_{12} \cdots\cdots\cdots c_{13}c_{13} \\ c_{21}c_{21} & c_{21}c_{22} \cdots\cdots\cdots c_{23}c_{23} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ c_{51}c_{51} & c_{51}c_{52} \cdots\cdots\cdots c_{53}c_{53} \end{pmatrix}$$

A number of r points $P_r = (x_{r_1}, x_{r_2}, x_{r_3})$ $r = 1, 2, 3, \ldots,$ are said to be *quadratically dependent* if the r rows

$$x_{r1}x_{r1} \quad x_{r1}x_{r2} \quad x_{r2}x_{r2} \quad x_{r1}x_{r3} \quad x_{r2}x_{r3} \quad x_{r3}x_{r3}$$

are linearly dependent, that is, if it is possible to find r constants $k_1, k_2, \ldots, k_r$, not all zero, such that the six equations

$$\begin{aligned} k_1\, x_{11}x_{11} + k_2\, x_{21}x_{21} + k_3\, x_{31}x_{31} + \cdots\cdots + k_r\, x_{r1}x_{r1} &= 0 \\ k_1\, x_{11}x_{12} + k_2\, x_{21}x_{22} + k_3\, x_{31}x_{32} + \cdots\cdots + k_r\, x_{r1}x_{r2} &= 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot& \\ k_1\, x_{13}x_{13} + k_2\, x_{23}x_{23} + k_3\, x_{33}x_{33} + \cdots\cdots + k_r\, x_{r3}x_{r3} &= 0 \end{aligned}$$

hold. In this case, we also say that any one of the r points is quadratically dependent on the remaining $r-1$ points. Otherwise, the r points P_r are *quadratically independent*. From this definition, it follows that the r points P_r are quadratically dependent if the rank of the following matrix is less than r

$$\begin{pmatrix} x_{11}x_{11} & x_{11}x_{12} & x_{12}x_{12} & x_{11}x_{13} & x_{12}x_{13} & x_{13}x_{13} \\ x_{21}x_{21} & x_{21}x_{22} & x_{22}x_{22} & x_{21}x_{23} & x_{22}x_{23} & x_{23}x_{23} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ x_{r1}x_{r1} & x_{r1}x_{r2} & x_{r2}x_{r2} & x_{r1}x_{r3} & x_{r2}x_{r3} & x_{r3}x_{r3} \end{pmatrix} \qquad (11.11)$$

Since the matrix has always six columns, its rank can never be greater than six and consequently is always less than r when r is greater than 6. Hence, if $r > 6$, the r points are always quadratically dependent.

If $r = 6$, the r points are quadratically dependent if the determinant $|x_{ri}\, x_{rj}| = 0$. So, *six points are quadratically independent if there exists no conic passing through them.*

If $r < 6$, it is always possible to find a point $Q = (x_1, x_2, x_3)$ quadratically independent of the r points P_r. For, in the matrix

$$\begin{pmatrix} x_{11}x_{11} & x_{11}x_{12} \ldots\ldots x_{13}x_{13} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_{r1}x_{r1} & x_{r1}x_{r2} \ldots\ldots x_{r3}x_{r3} \\ x_1x_1 & x_1x_2 \ \ldots\ldots x_3x_3 \end{pmatrix}$$

it is always possible to choose x_1, x_2, x_3 such that there is at least one $(r+1)$-rowed determinant which is not zero.

If $r = 5$, the r points are quadratically dependent if the rank of the matrix (11.11) is less than five. We may therefore state the result, obtained before, as follows :

Five quadratically independent points determine a conic uniquely.

The geometrical meaning of a point Q quadratically dependent on (or independent of) five quadratically independent points $P_1, P_2, \ldots P_5$ is that Q lies (or does not lie) on the conic through $P_1, P_2, \ldots P_5$. Through four given points which are quadratically independent there pass a system of conics. These conics cover the projective plane in such a manner that through any point which is quadratically independent of the four given points there passes just one of these conics.

We now consider the significance of quadratic dependence. It should be noticed at the outset that, by a transformation of projective coordinates, a quadratically dependent system of points is transformed into a quadratically dependent system of points.

(1) *A point quadratically dependent on a given point.* Let the coordinates of the given point be (1, 0, 0) and those of another point (a_1, a_2, a_3). If the two points are quadratically dependent, the rank of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a_1^2 & a_1a_2 & a_2^2 & a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}$$

must be one. So, all second order determinants must be zero, *i.e.*,

$$a_1a_2 = a_2^2 = a_1a_3 = a_2a_3 = a_3^2 = 0$$

These are satisfied if and only if $a_2 = a_3 = 0$. Therefore two quadratically dependent points are coincident.

(2) *A point quadratically dependent on two given points.* Let the coordinates of the two given points be (1, 0, 0), (0, 1, 0) and those of

another be (a_1, a_2, a_3). If the three points are quadratically dependent, the rank of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ a_1^2 & a_1a_2 & a_2^2 & a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}$$

must be less than three. So, all third order determinants must be zero, *i.e.*,

$$a_1a_2 = a_1a_3 = a_2a_3 = a_3^2 = 0.$$

or

$$a_3 = a_1a_2 = 0.$$

So, either $a_1 = a_3 = 0$ or $a_2 = a_3 = 0$.

Therefore the point (a_i) must coincide with one of the given points.

(3) *A point quadratically dependent on three given points.*

(*i*) Let the three given points be noncollinear and, without loss of generality, let their coordinates be (1, 0, 0), (0, 1, 0), (0, 0, 1) and those of another point be (a_1, a_2, a_3). If the four points are quadratically dependent, the rank of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a_1^2 & a_1a_2 & a_2^2 & a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}$$

must be less than four. So, all fourth order determinants must be zero, *i.e.*,

$$a_1a_2 = a_1a_3 = a_2a_3 = 0.$$

So, $a_1 = a_2 = 0$ or $a_2 = a_3 = 0$ or $a_3 = a_1 = 0$.

Therefore the point (a_i) must coincide with one of the given points.

(*ii*) Let the three given points be collinear. Without loss of generality, let their coordinates be (1, 0, 0), (0, 1, 0), (0, 0, 1). Then the rank of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ a_1^2 & a_1a_2 & a_2^2 & a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix}$$

must be less than four. So,

$$a_1a_3 = a_2a_3 = a_3^2 = 0, \quad \text{or} \quad a_3 = 0.$$

Therefore the point (a_i) must lie on the line joining the three given points. In other words, all points of a line are quadratically dependent.

(4) *A point quadratically dependent on four given points.*

(*i*) Let no three of the four given points be collinear and let their coordinates be (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1). Then the rank of the 5-rowed matrix, constructed in the same manner as in the previous cases, must be less than five. That is, the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ a_1a_2 & a_1a_3 & a_2a_3 \end{pmatrix}$$

must be less than two. Hence we must have

$$a_1a_2 = a_1a_3 = a_2a_3$$

So, $a_1 = a_2 = a_3$ or $a_1 = a_2 = 0$ or $a_1 = a_3 = 0$ or $a_2 = a_3 = 0$. Therefore the point (a_i) must coincide with one of the four given points.

(*ii*) Let three of the four given points be collinear and the coordinates of the four points be (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1), the first three being collinear. Then, it may be seen as in the case (*i*) above that the rank of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ a_1a_2 & a_1a_3 & a_2a_3 \end{pmatrix}$$

must be less than two. Hence we must have

$$a_1a_3 = a_2a_3 = 0.$$

So either $a_3 = 0$ or $a_1 = a_2 = 0$.

Therefore the point (a_i) must either lie on the line of three of the given points or must coincide with the remaining given point.

(5) *A point quadratically dependent on five given points.*

(*i*) If the five points are quadratically independent and the sixth point is dependent on them, then the rank of the matrix (11.11) is equal to five. That means that the coordinates of the sixth point must satisfy a quadratic equation in which every coefficient cannot be zero, *i.e.*, the sixth point lies on a conic uniquely defined by the five given points. On the other hand, the five given different points are quadratically independent if and only if no four of them are collinear, and six points on a conic are always quadratically dependent. Hence, *if no four of five given* *points are collinear, there exists one and only one conic passing through them.* If this conic is degenerate, three points must be collinear, and conversely.

(*ii*) If the five given points are quadratically dependent, the six points are also quadratically dependent. In this case, four points are collinear and there exist an infinity of (degenerate) conics passing through the five given points, at least one through every point of the plane.

41. Projective theory of conics. (I) *Projective generation of conics.* Suppose that we are given any three rays a, b, c of one pencil of lines $(abc\ldots\ldots)$ with centre S to correspond respectively to the three rays a', b', c' of another pencil $(a'b'c'\ldots\ldots)$ with centre S', which is supposed to be different from S. We can then establish, by geometrical construction, a definite projectivity between the pencils in which (a,a'), (b,b') (c,c') are pairs of corresponding rays. The construction has already been given in (2) § 29, and it need not be repeated here. The only thing we have to note is that there is here no distinction between an ordinary point and a point at infinity, an ordinary line and the line at infinity.

Referring to the construction and the property of the cross-ratio in determining a projectivity, the two fundamental results which should be emphasised are first, any three distinct rays of one pencil may be related to any three distinct rays of the other pencil by at most two perspectivities and secondly, a projectivity between two pencils is uniquely determined when three pairs of corresponding rays are given.

We now recall, what has been shown at the end of § 24, that the points of intersection of corresponding lines of particular projective pencils lie on certain conics. That idea shall now be generalised.

Consider the locus of the points in which the corresponding rays of two projective pencils intersect. If the two pencils are perspective, then the ray SS' of the pencil (S) corresponds to the ray $S'S$ of the pencil (S') ; hence every point of the line SS' belongs to the locus ; moreover, corresponding rays meet on a line (the axis of perspectivity). Thus the locus consists of two lines and is therefore a degenerate conic. To consider the general case, let us use analytic method. Three different rays a, b, c of the pencil (S) can always be represented by

$$L_1(x_1, x_2, x_3) = 0, \quad L_2(x_1, x_2, x_3) = 0, \quad pL_1(x_1, x_2, x_3) + qL_2(x_1, x_2, x_3) = 0.$$

where the functions L_1 and L_2 are linear in x_1, x_2, x_3 and $p \neq 0$, $q \neq 0$ are constants. Putting

$$pL_1(x_1, x_2, x_3) = l_1(x_1, x_2, x_3), \quad qL_2(x_1, x_2, x_3) = l_2(x_1, x_2, x_3),$$

we get the rays a, b, c, represented respectively by

$$l_1(x_1, x_2, x_3) = 0, \quad l_2(x_1, x_2, x_3) = 0, \quad l_1(x_1, x_2, x_3) + l_2(x_1, x_2, x_3) = 0.$$

An arbitrary ray d of the pencil (S) is then represented by

$\gamma l_1(x_1, x_2, x_3) + \lambda l_2(x_1, x_2, x_3) = 0$, and their cross ratio is $(ab, cd) = \dfrac{\gamma}{\lambda}$.

Let $a', b', c'\ d'$, be the rays of (S') which correspond to a, b, c, d; then $(a'b', c'd') = (ab, cd) = \dfrac{\gamma}{\lambda}$. We can now represent the rays $a'\ b'\ c'$ by

$$l_1'(x_1, x_2, x_3) = 0,\ l_2'(x_1, x_2, x_3) = 0,\ l_1'(x_1, x_2, x_3) + l_2'(x_1, x_2, x_3) = 0,$$

where l''s are also linear functions; then d' must be represented by

$$\gamma l_1'(x_1, x_2, x_3) + \lambda l_2 (x_1, x_2, x_3) = 0$$

The point of intersection of d and d' satisfies therefore the two equations

$$\gamma l_1(x_1, x_2, x_3) + \lambda l_2(x_1, x_2, x_3) = 0$$

$$\gamma l_1'(x_1, x_2, x_3) + \lambda l_2'(x_1, x_2, x_3) = 0$$

On the other hand, let (x_1, x_2, x_3) be a point in which two corresponding lines of the pencils meet. Then there exist values of γ, λ for which the above equations hold, and this is possible if and only if

$$\begin{vmatrix} l_1(x_1, x_2, x_3) & l_2(x_1, x_2, x_3) \\ l_1'(x_1, x_2, x_3) & l_2'(x_1, x_2, x_3) \end{vmatrix} = 0$$

This is an equation of the second degree. Hence the points of he locus must satisfy an equation of the second degree, and every solution (x_1, x_2, x_3) of this equation is a point of the locus. An equation of second degree either represents a conic or is an identity, every coefficient being zero. But since S and S' are supposed distinct, every point of the projective plane does not belong to the locus and the locus contains at least five points which are not situated on a line. Hence the locus is a conic which is either nondegenerate or is a pair of distinct lines We therefore state the following theorem :

If there is a projectivity connecting two nonconcentric pencils of lines, the locus of the points of intersection of the corresponding lines is a conic which is either nondegenerate or consists of a pair of lines.

Let now five points be given which are quadratically independent and let them be denoted in an arbitrary manner by S, S', A, B, C. If there are three collinear points among them, we will suppose that A, B, C are collinear and that neither S nor S' is situated on the line ABC. At any rate, the lines

$$SA = a, \quad SB = b, \quad SC = c, \quad S'A = a', S'B = b', S'C = c'$$

are all distinct. If the rays a, b, c of the pencil (S) correspond to the rays a', b', c' of (S'), then the locus of the points in which corresponding rays meet is a conic which obviously passes through A, B, C. The conic passes also through S, S' since the line SS' of the pencil (S) meets the corresponding line of the pencil (S') in the point S', and similarly for the point S. In the preceding article it has been proved that there exists only one such conic passing through these five points. Thus *every nondegenerate conic and every pair of distinct lines can be generated by two nonconcentric projective pencils.*

On the other hand, let S and S' be two arbitrary different points of a nondegenerate conic which are the centres of two pencils of lines. If we set up a correspondence between those rays of S and S' which intersect on the conic, then quadruplets of corresponding rays have the same cross-ratios. For, if A, B, C, D are four points of the conic, different from S and S', the locus generated by the pencils whose corresponding rays are SA, $S'A$; SB, $S'B$; SC, $S'C$ is a conic passing through S, S', A, B, C and is therefore identical with the given conic. Hence SD corresponds to $S'D$ and the following cross-ratios are equal, namely

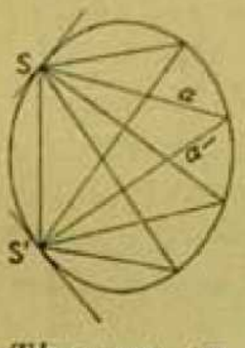
$$(SA\ SB,\ SC\ SD) = (S'A\ S'B,\ S'C\ S'D)$$

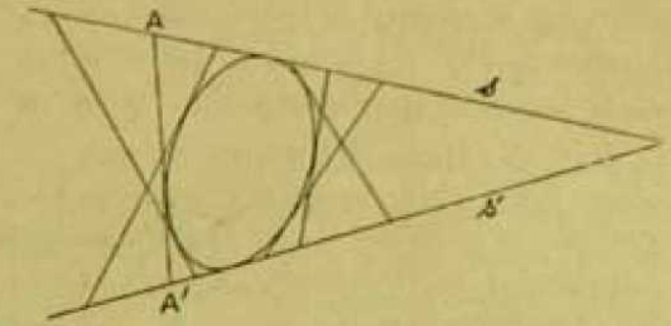
Suppose, for the moment, that the conic is a circle. Then $\measuredangle ASB = \measuredangle AS'B$ for every pair of points A, B of the circle, and the two pencils (S) and (S') are therefore congruent. Hence we may get an alternative proof of the projective generation of a nondegenerate conic by "generalisation by collineation," as in §36.

Consider dually two rows of points $(ABC..)$ and $(A'B'C'..)$ on different lines s and s' and establish a projectivity between the two rows, as in (1) § 29. The lines joining corresponding points of the rows generate an envelope. To investigate the nature of this envelope, we need only repeat the above investigation regarding two projective pencils, using always the dual terms, *i.e.*, the coordinates (x_1, x_2, x_3) of a point have to be replaced by the coordinates (u_1, u_2, u_3) of a line, and conversely. It would follow that the envelope is of the second class and is either nondegenerate or consists of two points. We therefore state the following theorem :

If there is a projectivity connecting two rows of points which are not cobasal, the envelope of the lines joining corresponding points is a conic which is either nondegenerate or consist of a pair of points.

Strictly speaking, a degenerate conic locus is a pair of rows of points and a degenerate conic envelope is a pair of pencils of lines. The conic locus and the conic envelope are also called the *point conic* and the *line conic* respectively. The diagram given below shows the conics when they are nondegenerate. To the rays SS' of (S) and (S') correspond respectively the tangents at S' and S. To the points ss' of (s) and (s') correspond respectively the points of contact on s' and s.

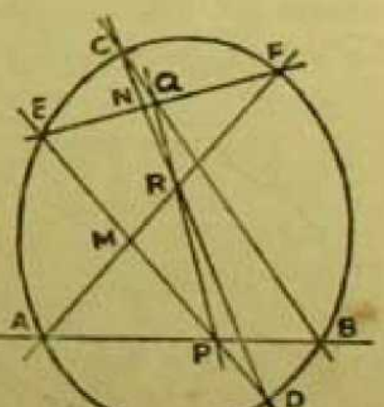




(II) *Theorems on conics and hexagons.* Let A, B, C, D, E, F be six points, no three of which are collinear. By 'the hexagon $ABCDEF$' we shall mean the figure obtained by drawing the lines AB, BC, CD, DE, EF, FA. These six lines are called the sides and the six points the vertices of the hexagon. Cyclical permutations of $ABCDEF$ and $FEDCBA$ (such as $CDEFAB$, $CBAFED$, etc.) represent the same hexagon and any other permutation represents a different hexagon. Hence, from the six vertices we obtain sixty different hexagons. The sides of the hexagon $ABCDEF$ are grouped into three pairs of opposite sides : AB, DE ; BC, EF ; CD, FA.

We have just seen that five points, no three of which are collinear, determine a nondegenerate conic uniquely. The following theorem gives the necessary and sufficient condition that six points should lie on the same conic.

PASCAL'S theorem. If the six vertices of a hexagon $ABCDEF$ lie on a conic, the points of intersection of the three pairs of opposite sides AB, DE ; BC, EF ; CD, FA are collinear. Conversely, if the hexagon is such that the three pairs of opposite sides intersect in three collinear points, the six vertices of the hexagon lie on a conic.



Proof. Let the points of intersection of AB, DE ; BC, EF and CD, FA be P, Q and R respectively. Also, let ED and AF intersect in M, CD and EF intersect in N. Suppose that the six points A, B, C, D, E, F are points of a conic. The pencils $A(EDBF\ldots.)$ and $C(EDBF\ldots.)$ with centres A and C respectively are projective, because the corresponding rays AE, CE, etc. intersect on the conic. Therefore the rows $(EDPM\ldots.)$ and $(ENQF\ldots.)$ in which these two pencils are cut by the lines

ED and EF respectively are projective. But these two projective rows have the point E as the self-corresponding point and hence the rows are perspective (§ 29). Therefore the lines DN, PQ, MF are concurrent; that is, the lines CD, PQ, AF are concurrent. So, the lines CD and AF intersect on the line PQ; that is, the point R lies on the line PQ.

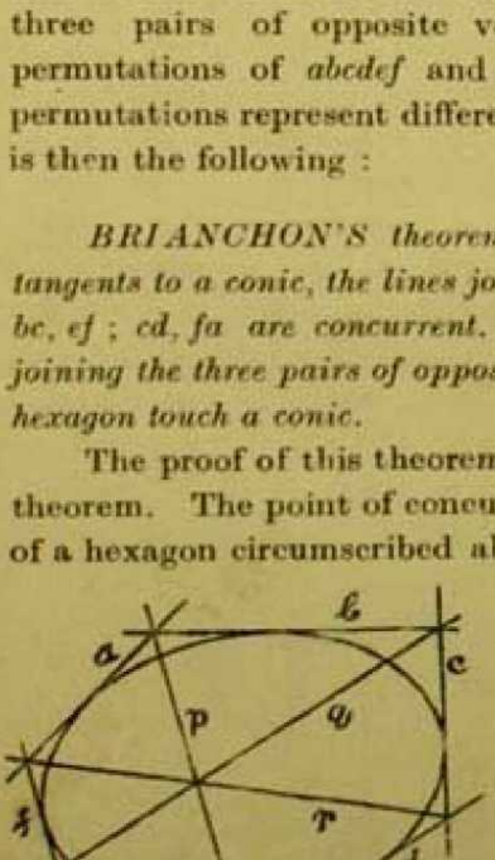
Conversely, suppose that the points P, Q, R are collinear. Now the five points A, B, C, D, E determine a conic. If the point F does not lie on this conic, let EF intersect the conic in the point F'. Also, let CD intersect AF' in R'. Since $ABCDEF'$ is a hexagon inscribed in the conic, the three points P, Q, R' are, by what has been proved above, collinear. But, by hypothesis, P, Q, R are collinear and so R and R' lie on PQ. Also, by construction, they lie on CD. Therefore R' must coincide with R and hence F' must coincide with F. Accordingly, the six points A, B, C, D, E, F lie on a conic.

The line PQR is known as a *Pascal's line*. From six points on a conic we obtain sixty Pascal's lines.

Let a, b, c, d, e, f be six lines, no three of which are concurrent. By 'the hexagon *abcdef*' we shall mean the figure whose sides are the six lines and whose vertices are the six points ab, bc, cd, de, ef, af. There are three pairs of opposite vertices : ab, de ; bc, ef and cd, fa. Cyclical permutations of *abcdef* and *fedcba* represent the same hexagon and other permutations represent different hexagons. The dual of Pascal's theorem is then the following :

BRIANCHON'S theorem. If the six sides of a hexagon abcdef are tangents to a conic, the lines joining the three pairs of opposite vertices ab, de ; bc, ef ; cd, fa are concurrent. Conversely, if the hexagon is such that the lines joining the three pairs of opposite vertices are concurrent, the six sides of the hexagon touch a conic.

The proof of this theorem is obtained by dualising the proof of Pascal's theorem. The point of concurrence of the lines joining the opposite vertices of a hexagon circumscribed about a conic is called a *Brianchon's point*. From six lines touching a conic we obtain sixty Brianchon's points.



The following are some of the interesting corollaries of the above two theorems which are obtained by supposing one or more pairs of vertices or sides of the hexagon to coincide.

Cor. (*i*) If A, B, C, D, E are five points on a conic, the point of intersection of the tangent at A and CD is collinear with the intersections of AB, DE and BC, EA. (Suppose $F = A$).

If a, b, c, d, e are five tangents to a conic, the line joining the point of contact of a and cd is concurrent with the lines ab, de, and bc, ea.

(*ii*) If A, B, C, D are four points on a conic, the point of intersection of the tangents at A and C is collinear with the intersections of AB, CD and BC, DA. (Suppose the hexagon is $AABCCD$).

If a, b, c, d, are four tangents to a conic, the line joining the points of contact of a and c is concurrent with lines ab, cd and bc, da.

(*iii*) If A, B, C, are three points on a conic and a, b, c are the tangents thereat, the points of intersection of a, BC ; b, CA and c, AB are collinear and the lines joining A, bc; B, ca and C, ab concurrent. (Suppose the hexagon is $AABBCC$).

We have all along been supposing that the conic is nondegenerate. In Pascal's theorem, let us suppose that the conic is degenerate and consists of the two lines ACE and BDF. It is then immediately seen that *Pappus' theorem is a particular case of Pascal's theorem.*

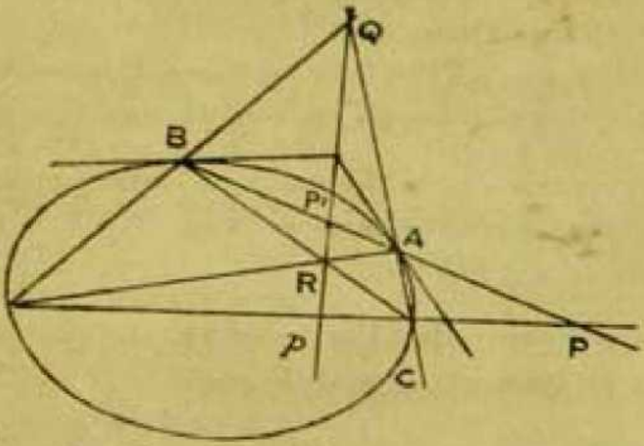
As an *application* of Pascal's theorem, suppose it is required to construct a conic through five given points, no three of which are collinear. The construction may be given as follows :

Let A, B, C, D, E, be the five points. Draw any line g through one of the points, A say, not passing through any of the remaining points. Let the point of intersection of AB, DE be P, of g, CD be R and of PR, BC be Q ; finally, let EQ meet g in F. Thus $ABCDEF$ is a hexagon such that the three pairs of opposite sides meet in three collinear points P, Q, R. Therefore F must lie on the conic through the five given points. Now, by taking different lines g through A we obtain different points F; and so, by constructing a sufficiently large number of points F, the conic can be constructed.

Similarly, given four points and a tangent at one of them or three points and the tangents at two of them, no three points being collinear and no point being situated on the tangent at a different given point, the conic can be constructed by the help of Pascal's theorem.

(III) *Synthetic treatment of pole and polar.* Take a conic and a point P not lying on the conic. Through P draw any two lines meeting the conic in A, B and C, D respectively. Let AC, BD meet in Q. AD,

BC meet in R. Finally, let the line QR be denoted by p. If the lines AB, p meet in P', then, from the complete quadrangle $QCRD$, the points P, P' are harmonically separated by A, B (§ 29). Similarly, if CD, p meet in P'', then P, P'' are harmonically separated by C, D. Therefore the line p is the polar of the point P with respect to the conic.



By Pascal's theorem Cor. (*ii*), the tangents to the conic at A and B intersect on p. Similarly, the tangents at C, D also intersect on p. Thus, if through P we draw any number of lines to meet the conic in distinct pairs of points (A, B), (C, D),....., the following points lie on the polar of P (points conjugate to P) : (*i*) the meet of AC, BD and the meet of AD, BC, (*ii*) the point on every line AB which is harmonically separated from P by the pair (A, B) and (*iii*) the meet of the tangents at every point pair (A, B).

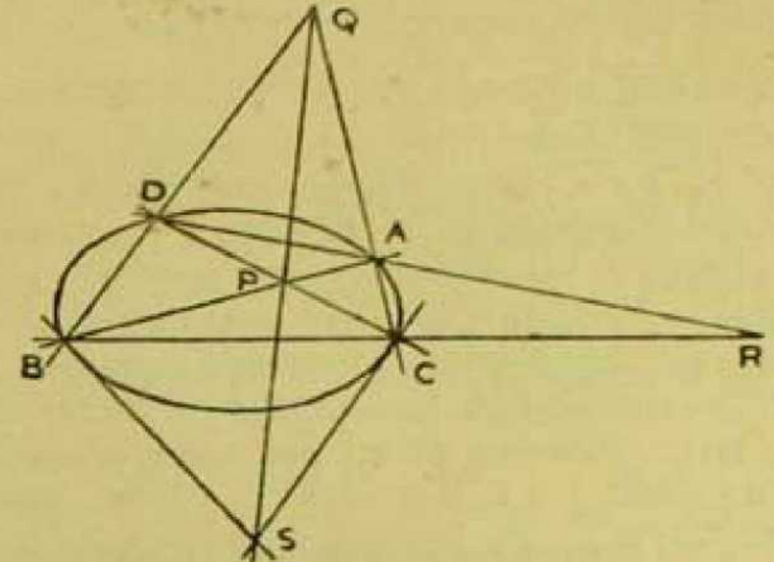
It follows that the polar of Q is the line PR and the polar of R is the line PQ. Therefore the triangle PQR is a polar triangle. But this triangle is the diagonal triangle of the complete quadrangle $ABCD$. Thus, *the diagonal triangle of a complete quadrangle whose vertices lie on a conic is a polar triangle with respect to the conic.* Conversely, every triangle which is polar triangle with respect to a conic can be considered as a diagonal triangle of a complete quadrangle whose vertices lie on the conic. If follows that all conics which pass through the same four points, no three of which are collinear, have a common polar triangle.

Applying the principle of duality, we take a line p which does not touch the conic and draw distinct pairs of tangents (a,b), (c, d),..... from points on p. Then the following lines pass through the pole of p (lines conjugate to p) : (*i*) the line joining ac, bd and the line joining ad, bc, (*ii*) the line through every point ab which is harmonically separated from p by the pair (a, b) and (*iii*) the line joining the points of contact of every tangent pair (a, b).

Also, *the diagonal triangle of a complete quadrilateral whose sides touch a conic is a polar triangle with respect to the conic.*

STAUDT'S theorem. Let the three vertices of a triangle ABC lie on a conic and S be the pole of BC with respect to the conic ; then any line through S meets AB and AC in conjugate points.

Proof. Let any line through S meet AB, AC in P, Q respectively and let CP meet the conic again in D; also, let BC and AD meet in R. Now, the four points A, B, C, D are on the conic; so, by Pascal's theorem or by the construction of pole and polar given above, the point of intersection of the tangents at B and C is collinear with the points of intersection of AB, DC and of BD, CA. Therefore, BD meets CA in Q. Hence PQR is the diagonal triangle of the complete quadrangle $BCAD$ and is therefore a polar triangle with respect to the conic. Accordingly, P and Q are conjugate points.



Applications. We now establish a few theorems which follow from the projective properties of conics given so far.

(1) *The complete quadrangle formed by four points on a conic and the complete quadrilateral formed by four tangents thereat have the same diagonal triangle.*

Let A, B, C, D be four points on a conic and a, b, c, d the tangents to the conic at these points respectively. Also, let the point of intersection of AB, CD be P, of BD, AC be Q, of AD, BC be R; the line joining ab, cd be e, joining bd, ac be f and joining ad, bc be g. Then, PQR is the diagonal triangle of the complete quadrangle $ABCD$ and efg the diagonal triangle of the complete quadrilateral $abcd$.

Now, by Pascal's theorem cor. (*ii*), the points P, R lie on f. Also, if the points A, B, C, D are taken in different orders then, from $ABDC$, the points P, Q lie on g, and from $ACBD$, the points Q, R lie on e. Hence the theorem.

(2) *If two complete quadrangles have the same set of diagonal points, their eight vertices lie on a conic.*

Let $ABCD$, $A'B'C'D'$ be two complete quadrangles and let the lines AB, CD, $A'B'$, $C'D'$ meet in P; BC, AD, $B'C'$, $A'D'$ meet in Q; AC, BD, $A'C'$, $B'D'$ meet in R. Also, let PA' meet QR in E. Now, the five points A, B, C, D, A' determine a conic Γ. Since PQR is the diagonal triangle of the complete quadrangle $A'B'C'D'$, the points P, E

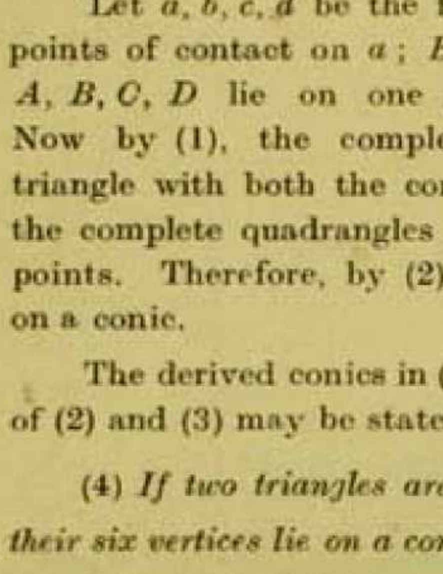
are harmonically separated by the points A', B'. But PQR is a polar triangle of the conic Γ and A' is a point on Γ; therefore B' must also be on Γ. Similarly C', D' must also be on Γ. Hence the theorem.

(3) *If two conics are inscribed in the same quadrilateral, the eight points of contact lie on a conic.*

Let a, b, c, d be the four sides of the quadrilateral and A, A' the points of contact on a; B, B' on b; C, C' on c; D, D' on d; the points A, B, C, D lie on one conic and A', B', C', D' lie on the other. Now by (1), the complete quadrilateral *abcd* has the common diagonal triangle with both the complete quadrangles $ABCD$, $A'B'C'D'$. Hence the complete quadrangles $ABCD$, $A'B'C'D'$ have the same set of diagonal points. Therefore, by (2), the eight points $A, B, C, D, A', B', C', D'$ lie on a conic.

The derived conics in (2) and (3) may be line pairs. The dual theorems of (2) and (3) may be stated and proved in the same way by dualising.

(4) *If two triangles are both self-polar with respect to a given conic, their six vertices lie on a conic and their six sides touch a conic.*



Let ABC, PQR be two triangles self-polar with respect to a conic. Let BC meet PQ and PR in Q' and R'; QR meet AB and AC in B' and C'. Now, Q' is the meet of PQ and BC whose poles are R and A; therefore the polar of Q' is AR. Similarly, the polar of R' is AQ. Hence, we have the cross-ratios

$$A(BC, QR) = (CB, R'Q')$$

(because, by § 27.1, the cross-ratios of polars and corresponding poles are equal)

$$= P(CB, R'Q), \text{ by projection from } P,$$

$$= P(BC, Q'R') = P(BC, QR)$$

So, $$A(BC, QR) = P(BC, QR).$$

So, by (I) of this article, the six points A, B, C, P, Q, R lie on a conic. Similarly,

$$(BC, Q'R') = A(CB, RQ) = A(C'B', RQ) = (C'B', RQ) = (BC', QR)$$

So, $$(BC\ Q'R') = (B'C', QR).$$

So, by (I) the four lines BB', CC' QQ' RR' touch a conic which is also touched by the bases BC, QR of the projective rows. But the four lines are respectively AB, AC, PQ, PR. So, the sides touch a conic.

By the theory of pole and polar, a given conic Γ sets up a polar correlation (P, p) in the plane. To a line PQ corresponds the point pq and if P, q are conjoint, then p, Q are also conjoint. Thus the principle of duality may be justified in this way.

Suppose a conic Γ_1 is generated by two projective pencils. Then the polar reciprocal (§ 22) of Γ_1 with respect to Γ is a conic Γ_2 generated by two projective rows, the bases of the two rows being the polars of the centres of the two pencils with respect to Γ. Let the polars of two points P, Q with respect to Γ be p, q. We then have the following properties :

(*i*) If P, q are pole and polar with respect to Γ_1, then p, Q are polar and pole with respect to Γ_2.

(*ii*) If P, Q are conjugate points with respect to Γ_1, then p, q are conjugate lines with respect to Γ_2.

From these properties the following theorem may be deduced :

(5) *Any two conics and the polar reciprocal of one with respect to the other have a common self-polar triangle.*

(IV) *DESARGUES' theorem on involution. If K, L, M, N are four points on a conic, any transversal cuts the conic and the pairs of opposite sides of the complete quadrangle $KLMN$ in pairs of conjugate points of an involution.*

Proof. Let a transversal u meet the conic in P, P' and the three pairs of opposite sides KL, MN ; KN, LM ; KM, LN of the complete quadrangle $KLMN$ in the three pairs of points (A, A'), (B, B'), (C, C'). Now, since the six points K, L, M, N, P, P' are on the conic,

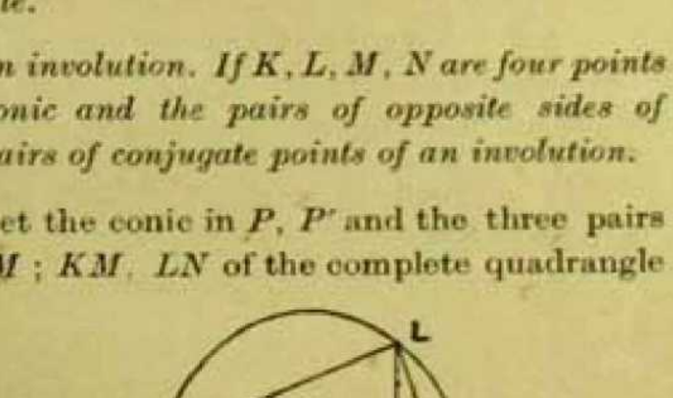
$$K(LN, PP') = M(LN, PP')$$

or $\quad (AB, PP') = (B'A', PP')$

(by section by u) $= (A'B', P'P)$

So, $\quad (AB, PP') = (A'B', P'P)$

In this projective correspondence, the points P, P' correspond to one another doubly. Therefore (A, A'), (B, B'), (P, P') are pairs of conjugate points of an involution. Similarly, it may be seen that (B, B'), (C, C'), (P, P') are pairs of conjugate points of an involution. But two pairs (B, B'), (P, P') determine an involution uniquely. Hence (A, A') (B, B') (C, C'), (P, P') are pairs of conjugate points of an involution.



The dual theorem may be stated thus :

If k, l, m, n are four tangents to a conic, the two tangents to the conic from any point and the lines joining the same point to the three pairs of opposite vertices of the complete quadrilateral klmn are pairs of conjugate lines of an involution.

The proof of this theorem is obtained by dualising the proof given above.

The set of all conics passing through four points form a pencil of conics and the set of all conics touching four lines form a *range of conics.*

Let K, L, M, N be four points forming a quadrangle and u a line not passing through any of these points. In the pencil of conics through K, L, M, N there are either two conics which touch u or none at all. For, the pencil of conics determine an involution on u. If this involution is hyperbolic, there are two double points P and Q, say ; then the conic of the pencil which touches u at P is one of the conics while that which touches u at Q is the other. If the involution is elliptic, no conic of the pencil can touch u. Dually, in the range of conics touching four lines k, l, m, n, forming a quadrilateral, there are either two conics which pass through a given point, not situated on any of the four lines, or none at all. Strictly speaking, a pencil of conics is a pencil of conic loci and a range of conics is a pencil of conic envelopes.

42. Pencil of conics. *Pencil of conic loci.* The set of all conics defined by,

$$\gamma\phi_1+\lambda\phi_2 = 0, \tag{11.12}$$

where $\phi_1 \equiv \sum a_{ij}x_ix_j = 0$, $\phi_2 \equiv \sum b_{ij}x_ix_j = 0$, $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$ are two given conics and γ, λ take all values, excepting both zero, is said to form a pencil of conic loci. Two conics of the pencil $\gamma\phi_1+\lambda\phi_2 = 0$ and $\gamma'\phi_1+\lambda'\phi_2 = 0$ are distinct if $\gamma\lambda'-\gamma'\lambda\neq 0$. Any conic of the pencil is linearly dependent on $\phi_1 = 0$ and $\phi_2 = 0$ and is therefore linearly dependent on any two distinct conics of the pencil. And, since the pencil is known when ϕ_1 and ϕ_2 are known, $\phi_1 = 0$ and $\phi_2 = 0$ may be called the *base conics.* In fact, any two distinct conics of the pencil may be taken as base conics. All conics of the pencil pass through the points of intersection, real or without real trace, of the base conics.

Let (x'_i) be any given point and let

$$\mu_1 = \sum a_{ij}x'_ix'_j, \quad \mu_2 = \sum b_{ij}x'_ix'_j$$

If a conic of the pencil passes through (x'_i), γ/λ must satisfy $\gamma\mu_1+\lambda\mu_2 = 0$. Hence, *through a given point there passes just one conic of the pencil unless*

$\mu_1 = \mu_2 = 0$. If $\mu_1 = \mu_2 = 0$ the given point is a point of intersection of the conics $\phi_1 = 0$, $\phi_2 = 0$, and all conics of the pencil pass through the given point. So, if the conics $\phi_1 = 0$, $\phi_2 = 0$ do not intersect in any real point, there is just one conic of the pencil which passes through the given point.

Let two points (x_i), (x'_i) be conjugate with respect to both the conics $\phi_1 = 0$, $\phi_2 = 0$. So

$$\sum a_{ij} x_i x'_j = 0, \quad \sum b_{ij} x_i x'_j = 0$$

Each of the two conics determines an involution of points formed by pairs of conjugate points on any line. If the involutions determined by both the conics on a line, say $x_1 = 0$, are the same, the two equations

$$a_{22} x_2 x_2' + a_{23}(x_2 x_3' + x_3 x_2') + a_{33} x_3 x_3' = 0$$

$$b_{22} x_2 x_2' + b_{23}(x_2 x_3' + x_3 x_2') + b_{33} x_3 x_3' = 0$$

must be the same. So, the rank of the matrix

$$\begin{pmatrix} a_{22} & a_{23} & a_{33} \\ b_{22} & b_{23} & b_{33} \end{pmatrix}$$

must be one. Now, the rank of the matrix remains unaltered if we replace b_{ij} by $\gamma a_{ij} + \lambda b_{ij}$. Hence, *if two conics of a pencil generate the same involution on a line, then every conic of the pencil generates the same involution on the same line.*

Let us, for the moment, suppose that our coordinates are special projective coordinates, *i.e.*, homogeneous Cartesian coordinates, and let us consider a circle

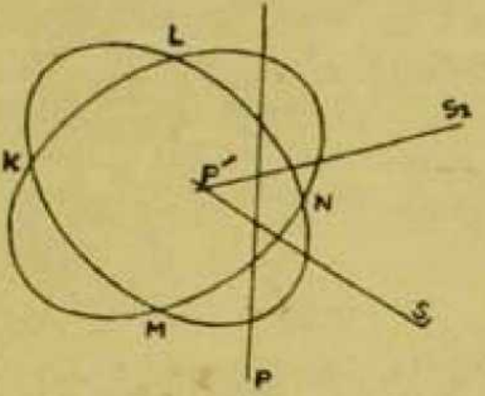
$$x_1^2 + x_2^2 + 2dx_1x_3 + 2ex_2x_3 + fx_3^2 = 0$$

The involution generated by the circle on the line at infinity $x_3 = 0$ is given by the equations

$$x_1 x_1' + x_2 x_2' = 0, \; x_3 = x_3' = 0$$

Since this is independent of the constants d, e, f, any other circle will generate the same involution on the line at infinity. Conversely, every conic generating this involution is a circle. For, in the equation of the involution we would have the coeff. of $x_1 x'_1$ = the coeff. of $x_2 x'_2$ and the coeff. of $(x_1 x'_2 + x_2 x'_1) = 0$.

Let all conics of the pencil (11.12) pass through four points K, L, M, N and u be a fixed line. Also, let P be a variable point on u and P' the point conjugate to P with respect to two (and therefore all) conics of the pencil. If S_1 and S_2 are the poles of u with respect to two arbitrary conics of the pencil, then the lines S_1P' and S_2P' are the polars of P with respect to these two conics. Therefore, the pencils of lines $S_1(P')$ and $S_2(P')$ with centres S_1 and S_2 are both projective to the row of points (P) on u (§ 27.1) and are therefore projective to one another. Hence, by (I) of the last article, *the locus of P' is a conic*, Γ say, passing through $S_1, S_2, \ldots$.



Let KL meet u in H and let H_1 be the harmonic conjugate of H with respect to K, L. So, the conic Γ passes through the six such points H_1 on the six sides of the complete quadrangle $KLMN$. Again let E, F, G be the diagonal points of the complete quadrangle $KLMN$ and let G' be the point of intersection of EF and u. Then, since EF is the polar of G with respect to all conics of the pencil, G, G' are conjugate points with respect to all these conics. So, Γ passes through the point G and, for the same reason, through E, F. Finally, if the conics of the pencil meet u in the pairs of points (A, A'), (B, B'), ..., then, by Desargues' theorem of the last article, these pairs of points form an involution on u. If the involution is hyperbolic and U, V are the double points of this involution, Γ passes through the points U, V. Thus the conic Γ passes through the nine (eleven) fixed points, H_1, E, F, G, (U, V).

Let us, for the moment, suppose that the line u is the line at infinity. So, the points S_1, S_2, ... are the centres of the conics of the pencil. Hence Γ is the locus of these centres. Further, the point H_1 is now the middle point of the segment KL and, as before, E, F, G are the diagonal points of the complete quadrangle $KLMN$. The locus Γ, of the centres, which passes through these nine fixed points is accordingly called the *nine-point conic*.

Degenerate conics of the pencil. In the pencil of conics (11.12) suppose that one at least of the conics is nondegenerate, say $\phi_1 = 0$ is nondegenerate, *i.e.*, $| a_{ij} | \neq 0$. If then there is any degenerate conic in the pencil, $\lambda \neq 0$; and for these degenerate conics γ/λ should be such as to make the determinant

$$| (\gamma/\lambda)a_{ij} + b_{ij} | = 0$$

This is a cubic equation in γ/λ. Hence, *there are not more than three and not less than one distinct degenerate conic.* As before, let the conics pass through the four points K, L, M, N. The different degenerate cases that may arise are the following :

(*a*) If the four points are distinct, there are three distinct degenerate conics, each consisting of a pair of opposite sides of the complete quadrangle $KLMN$.

(*b*) If two of the points are coincident, say K, L coincident at K, all conics have simple contact at K. There are here two distinct degenerate conics, one consisting of the tangent to all the conics at K and the line MN, and the other consisting of the lines KM, KN.

(*c*) If two pairs of points are coincident, say K, L coincident at K and M, N coincident at M, all conics have two simple contacts, one at K and the other at M or, as we say, double contact at K and M. There are here also two distinct degenerate conics, one consisting of the tangents at K and M and the other consisting of the line KM counted twice.

(*d*) If three of the points are coincident, say K, L, M coincident at K, all conics have three-point contact at K. There is here one distinct degenerate conic consisting of the common tangent at K and the line KN.

(*e*) If the four points are coincident at K, the conics have four-point contact at K. There is here also one distinct degenerate conic consisting of the common tangent at K counted twice.

Singular points. A point on a conic is said to be a *singular point* of the conic if every line determined by it and any other point of the conic is contained in the conic. It is evident that only degenerate conics have singular points. If the degenerate conic consists of two distinct lines, the point of intersection of these lines is the only singular point ; and if the degenerate conic consists of two coincident lines, every point of the line is a singular point.

Let (r_i) be a given point and (x_i) any other point of a conic $\sum c_{ij}x_ix_j = 0$. An arbitrary point $(\mu r_i + \nu x_i)$ of the line joining the two points is a point of the conic if

$$\mu^2\sum c_{ij}r_ir_j + 2\mu\nu\sum c_{ij}r_ix_j + \nu^2\sum c_{ij}x_ix_j = 0$$

The first and the last terms are, by hypothesis, zero ; so the condition reduces to $\sum c_{ij}r_ix_j = 0$. But as (x_i) is any point on the conic, the coefficients of x_1, x_2, x_3 must be separately zero, *i e.*

$$\sum_i c_{ij}r_i = 0, \quad j = 1, 2, 3 \qquad (11.13)$$

Thus, (11.13) is the condition that (r_i) is a singular point of the conic.

Now, we have seen that the degenerate conics of the pencil (11.12) will be given by those values of γ, λ as satisfy

$$| \gamma a_{ij} + \lambda b_{ij} | = 0,$$

where, as before, we suppose that $| a_{ij} | \neq 0$. For such values of γ, λ, there must exist quantities r_1, r_2, r_3 satisfying the three equations

$$\sum_i (\gamma a_{ij} + \lambda b_{ij}) r_i = 0, \quad j = 1, 2, 3 \qquad (11.13)$$

It therefore follows from (11.13) that (r_i) is a singular point of a degenerate conic of the pencil.

Further, for arbitrary values of γ, λ, the polar of a point (r_i) with respect to the conics of the pencil are

$$\sum_{ij} (\gamma a_{ij} + \lambda b_{ij}) r_i x_j = 0$$

These polars are the same for all conics of the pencil if γ, λ exist such that (11.13′) is satisfied, that is, if (r_i) is a singular point of a degenerate conic of the pencil. Hence, *the singular points of the degenerate conics of a pencil are the only points whose polars with respect to all conics of the pencil are the same.* It may be noted that the polar of a point with respect to a conic is undefined only when the conic is degenerate and the point is a singular point of the conic.

Equations of pencils. Consider a point and a line which are pole and polar with respect to all the conics of the pencil (11.12); the point is therefore a singular point of a degenerate conic of the pencil. The polarity determined by an arbitrary conic of the pencil is

$$\sigma u_i = \sum_j (\gamma a_{ij} + \lambda b_{ij}) x_j, \quad i = 1, 2, 3$$

For pole and polar, two cases may arise : (I) the pole conjoint with its polar and (II) the pole disjoint with its polar.

(I) Let the coordinates of the pole and the polar be (0, 0, 1) and (1, 0, 0) respectively. Since this polarity is to be satisfied by an arbitrary conic of the pencil, we must have

$$a_{23} = b_{23} = a_{33} = b_{33} = 0$$

Therefore the discriminant

$$| \gamma a_{ij} + \lambda b_{ij} | = (\gamma a_{13} + \lambda b_{13})^2 (\gamma a_{22} + \lambda b_{22})$$

For the degenerate conics of the pencil, we shall have this discriminant equal to zero. Two cases may again arise : (1) both the factors of the discriminant are zero for the same value of γ/λ and (2) there are two values of γ/λ for each of which the discriminant is zero.

(1) Here
$$-\gamma/\lambda = b_{13}/a_{13} = b_{22}/a_{22} = \rho \text{ (say)}$$

Since there is this one value for γ/λ, there is just one degenerate conic, and we take it as one of the base conics. As

$$a_{23} = a_{33} = b_{23} = b_{33} = 0, \quad b_{13} = \rho a_{13}, \quad b_{22} = \rho a_{22},$$

the equation of the pencil (11.12) can be written as

$$\gamma(a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a'_{13}x_1x_3 + a'_{22}x^2) + \lambda(b_{11}x_1^2 + 2b_{12}x_1x_2) = 0$$

Two sub-cases arise :

(*i*) $b_{12} = 0$. The equation of the pencil now takes the *normal* form

$$\gamma(ax_1^2 + 2bx_1x_2 + 2cx_1x_3 + dx_2^2) + \lambda x_1^2 = 0 ; \tag{11.14}$$

and the equation of the degenerate conic is now $x_1^2 = 0$.

(*ii*) $b_{12} \neq 0$. We apply the collineation

$$\sigma x'_1 = x_1, \quad \sigma x'_2 = b_{11}x_1 + 2b_{12}x_2, \quad \sigma x'_3 = x_3.$$

The equation of the pencil now reduces to the *normal* form (dropping the dashes)

$$\gamma(ax^2_1 + 2bx_1x_2 + 2cx_1x_3 + dx^2_2) + \lambda x_1x_2 = 0 ; \tag{11.15}$$

and the equation of the degenerate conic is now $x_1x_2 = 0$.

(2) Here $-\gamma/\lambda$ has two values b_{13}/a_{13} and b_{22}/a_{22} Let

$$b_{13}/a_{13} = \rho_1, \quad b_{22}/a_{22} = \rho_2$$

Since there are these two values of γ/λ, there are two degenerate conics, and we take them as the base conics. As

$$a_{23} = a_{33} = b_{23} = b_{33} = 0, \quad b_{13} = \rho_1 a_{13}, \quad b_{22} = \rho_2 a_{22},$$

the equation of the pencil (11.12) can be written as

$$\gamma(a_{11}x^2_1 + 2a_{12}x_1x_2 + 2a'_{13}x_1x_3) + \lambda(b_{11}x^2_1 + 2b_{12}x_1x_2 + b'_{22}x^2_2) = 0, \quad a'_{13}b'_{22} \neq 0$$

Apply the collineation

$$\sigma x'_1 = x_1, \quad \sigma x'_2 = (b_{12}/b'_{22})\, x_1 + x_2, \quad \sigma x'_3 = a_{11}x_1 + 2a_{12}x_2 + 2a'_{13}x_3$$

The equation then reduces to the form

$$\gamma x'_1x'_3 + \lambda(ax'^2_1 + bx'^2_2) = 0, \quad b = b'_{22} \neq 0$$

Finally, apply the collineation

$$\sigma x''_1 = \sqrt{|a|}\, x'_1, \quad \sigma x''_2 = \sqrt{|b|}\, x'_2, \quad \sigma x''_3 = x'_3$$

The equation of the pencil now takes one of the three *normal* forms (dropping the dashes)

$$\gamma x_1x_2 + \lambda(\epsilon x_1^2 + x_2^2) = 0, \quad \epsilon = 1, 0, -1 ; \tag{11.16}$$

and the equations of the two degenerate conics are now $x_1x_2 = 0$ and $\epsilon x_1^2 + x_2^2 = 0$, ϵ having the corresponding value.

(II) Let the coordinates of the pole and the polar, which are now supposed to be disjoint, be (0, 0, 1), and (0, 0, 1). Since this polarity is to be satisfied by an arbitrary conic of the pencil, we have

$$a_{13} = b_{13} = a_{23} = b_{23} = 0.$$

A degenerate conic of the pencil may be obtained by supposing $b_{33} = 0$, as this makes $| b_{ij} | = 0$, and let us take this conic as a base conic. When $b_{33} = 0$, we must have $a_{33} \neq 0$, as otherwise it would make the discriminant of the equation of the pencil zero. Without loss of generality we may assume $a_{33} = -1$.

The equation of the pencil (11.12) can now be written as

$$\gamma(a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 - x_3^2) + \lambda(b_{11}x_1^2 + 2b_{12}x_1x_2 + b_{22}x_2^2) = 0$$

As before, the coefficient of λ can, by the application of a suitable collineation, be reduced to $x_1^2 + \epsilon x_2^2$, $\epsilon = 1, 0, -1$.
So, the equation reduces to the form

$$\gamma(c_{11}x_1^2 + 2c_{12}x_1x_2 + c_{22}x_2^2 - x_3^2) + \lambda(x_1^2 + \epsilon x_2^2) = 0$$

There are three cases according as $\epsilon = 1, 0, -1$. Taking $\epsilon = 1$ and applying an orthogonal transformation in x_1, x_2 (a particular case of collineation), the coefficient of $x_1 x_2$ may be made to vanish while $x_1^2 + x_2^2$ remains invariant. The equation of the pencil then takes the form

$$\gamma(ax_1^2 + bx_2^2 - x_3^2) + \lambda(x_1^2 + x_2^2) = 0 \tag{11.17}$$

Similarly when $\epsilon = 0$, the equation takes the form

$$\gamma(ax_1^2 + bx_2^2 - x_3^2) + \lambda x_1^2 = 0 \tag{11.18}$$

When $\epsilon = -1$, the equation, by suitable change of coordinates, can be put in the form

$$\gamma(ax_1^2 + 2bx_1x_2 + cx_2^2 - x_3^2) + \lambda x_1x_2 = 0 \tag{11.19}$$

There are various subcases for each of the equations (11.17), (11.18), (11.19) and the equations can be put in simpler *normal* forms.

From the point of view of singular points, pencils of conics can be of the following five types : (*a*) pencil having three singular points, when there are three degenerate conics, (*b*) pencil having two singular points, when there are two degenerate conics each consisting of a pair of lines, (*c*) pencil having one singular point, when there is one degenerate conic consisting of a pair of lines, (*d*) pencil having a row of singular points, when there is one degenerate conic consisting of two coincident lines and (*e*) pencil having one and a row of singular points, when there are two degenerate conics, one consisting of a pair of lines and the other consisting of two coincident lines.

We have hitherto been dealing with conic loci. We may dualise all that we have said and obtain properties of a pencil of conic envelopes. When a degenerate conic envelope consists of two distinct points, the line joining these points is the singular line of the conic and when it consists of two coincident points, every line passing through the point is a singular line. We state briefly some of the dual properties.

Pencil of conic envelopes. The set of conics given by

$$\chi\psi_1 + \xi\psi_2 = 0, \tag{11.20}$$

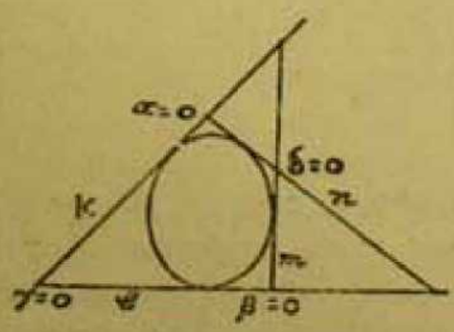
which are linearly dependent on two distinct conics

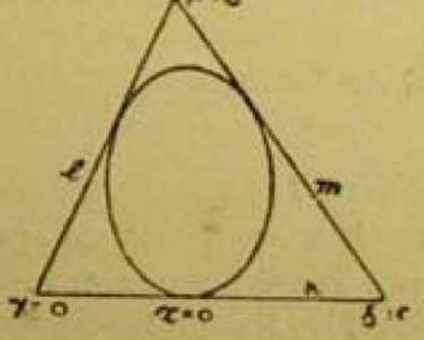
$$\psi_1 \equiv \sum a_{ij}u_iu_j = 0, \quad \psi_2 \equiv \sum b_{ij}u_iu_j = 0,$$

is said to form a *pencil of conic envelopes.* As in the case of a pencil of conic loci, a pencil of conic envelopes possesses the following properties : (1) Any two distinct conics of the pencil may be taken as the base conics, (2) there is a unique conic of the pencil which is tangent to a given line (*i.e.*, contains a given line as a line of envelope) other than the tangents common to all conics of the pencil, (3) if any two conics of a pencil generate the same involution of lines through a point, every conic of the pencil generates the same involution through the same point and (4) there are not more than three and not less than one distinct degenerate conics in the pencil.

Let the conics of the pencil (11.20) be tangents to four lines k, l, m, n (*i.e.*, the four lines are lines of envelope of all conics) and let the equations of the points kn, lm, kl, mn be $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ respectively.

(*a*) When the four lines are distinct, there are three distinct degenerate





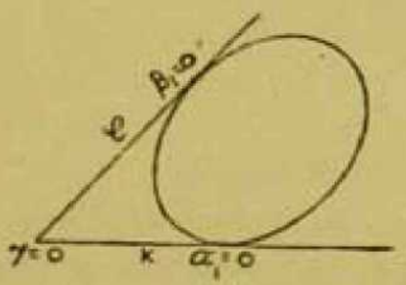
conics, each consisting of a pair of opposite vertices of the complete quadrilateral *klmn*. The equation of the pencil can be written as

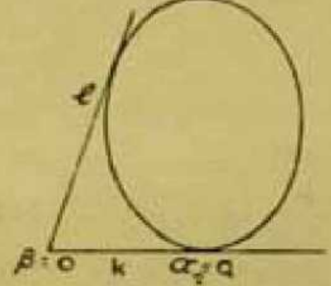
$$\chi\alpha\beta+\xi\gamma\delta=0$$

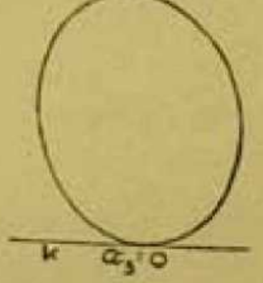
(*b*) When two of the lines are coincident, say *k*, *n* coincident with *k*, all conics have simple contact at the same point on *k*. If the equation of this point is $\alpha_1=0$, the two distinct degenerate conics are $\alpha_1\beta=0$ and $\gamma\delta=0$. The equation of the pencil can therefore be written as

$$\chi\alpha_1\beta+\xi\gamma\delta=0$$

(*c*) When two pairs of lines are coincident, say *k*, *n* coincident with *k* and *l*, *m* coincident with *l*, all conics have two simple contacts at two points on *k* and *l*, one on each. If the equations of this two points are $\alpha_1=0$, and $\beta_1=0$, the two distinct degenerate conics are $\alpha_1\beta_1=0$ and $\gamma^2=0$. The equation of the pencil can therefore be written as







$$\chi\alpha_1\beta_1+\xi\gamma^2=0$$

(*d*) When three of the lines are coincident, say *k*, *m*, *n* coincident with *k*, all coincs have three-point contact at the same point on *k*. If the equation of this point is $\alpha_2=0$, the one distinct degenerate conic is $\alpha_2\beta=0$. If $\psi=0$ be the equation of one of the nondegenerate conics of the pencil, the equation of the pencil can be written as

$$\chi\psi+\xi\alpha_2\beta=0$$

(*e*) When all the four lines are coincident with *k*, all the conics have four-point contact at the same point on *k*. If the equation of this point is $\alpha_3=0$, the one distinct degenerate conic is $\alpha_3{}^2=0$. So, if the equation of a nondegenerate conic of the pencil is $\psi=0$, the equation of the pencil can be written as

$$\chi\psi+\xi\alpha_3{}^2=0$$

Confocal conics. In connection with the theory of the pencil of conics, it may be advantageous to consider some properties of confocal conics. For this purpose, suppose that the equation (11.17) of the pencil of conic loci is given in homogeneous Cartesian coordinates. The equation of the pencil of conic envelopes of the same type will be given by an equation of the form

$$\chi(au_1^2+bu_2^2-u_3^2)+\xi(u_1^2+u_2^2) = 0$$

or

$$(au_1^2+bu_2^2-u_3^2)-\rho(u_1^2+u_2^2) = 0,$$

where ρ is an arbitrary constant. In point coordinates, this equation becomes

$$\begin{vmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & a-\rho & 0 & 0 \\ x_2 & 0 & b-\rho & 0 \\ x_3 & 0 & 0 & -1 \end{vmatrix} = 0$$

or

$$\frac{x_1^2}{a-\rho}+\frac{x_2^2}{b-\rho}-x_3^2 = 0$$

In nonhomogeneous coordinates we get

$$\frac{x^2}{a-\rho}+\frac{y^2}{b-\rho}-1 = 0 \qquad (11.21)$$

Let us interpret this equation geometrically. We have seen in § 14.1 that a focus of a conic is the point of intersection of two nonparallel isotropic tangents to the conic. In the case of a central conic we have four foci; and so a system of confocal conics consists of the nondegenerate conics of a pencil of conic envelopes which touch four isotropic lines, two of each kind (*i.e.* the four isotropic lines are lines of envelope of all these conics).

Let the coordinates of two real foci F, F' be $(\pm c, 0)$; then those of the two imaginary foci G, G' are $(0, \pm ic)$, $i^2=-1$. Therefore the equations of F, F', G, G' in nonhomogeneous line coordinates are

$$\alpha \equiv cu+1 = 0, \quad \beta \equiv cu-1 = 0$$

$$\gamma \equiv icv+1 = 0, \quad \delta \equiv icv-1 = 0$$

Hence the equation of the pencil in nonhomogeneous line coordinates is

$$\mu\alpha\beta - \nu\gamma\delta = 0, \quad \text{or} \quad \mu(c^2u^2-1)+\nu(c^2v^2+1) = 0$$

or

$$\frac{c^2\mu}{\mu-\nu}u^2+\frac{c^2\nu}{\mu-\nu}v^2 = 1, \quad \mu\nu(\mu-\nu) \neq 0,$$

where μ, ν are arbitrary constants. In point] coordinates, the equation becomes

$$\frac{\mu-\nu}{c^2\mu}x^2+\frac{\mu-\nu}{c^2\nu}y^2 = 1$$

Put $\dfrac{c^2\mu}{\mu-\nu} = a-\rho$. So, $\dfrac{c^2\nu}{\mu-\nu} = a-\rho-c^2 = b-\rho$ (say).

The above equation therefore becomes (11.21) where, since $b = a-c^2$, $a > b$. Hence the equation of a pencil of confocal conics is (11.21). For values of ρ lying between a and b, the equation represents confocal hyperbolas ; for $\rho < b$, the equation represents confocal ellipses ; and for $\rho > a$, the equation represents confocal conics without real trace. A system of confocal parabolas, in line coordinates, is a pencil of conic envelopes of the type (b) above.

42.1. Invariants of two conics. As in the last article, consider the conics

$$\phi_1 \equiv \sum a_{ij}x_ix_j = 0, \quad \phi_2 \equiv \sum b_{ij}x_ix_j = 0,$$

$$\psi_1 \equiv \sum A_{ij}u_iu_j = 0, \quad \psi_2 \equiv \sum B_{ij}u_iu_j = 0,$$

and $$\sum(\gamma a_{ij}+\lambda b_{ij})\, x_ix_j = 0,$$

where A_{ij}, B_{ij} are the cofactors of a_{ij}, b_{ij} in $| a_{ij} |$, $| b_{ij} |$ respectively. The conic loci $\phi_1 = 0$, $\phi_2 = 0$ are respectively equivalent to the conic envelopes $\psi_1 = 0$ and $\psi_2 = 0$.

Expanding the discriminant of the last equation as a cubic in γ and λ, it may be seen that

$$| \gamma a_{ij}+\lambda b_{ij} | = | a_{ij} | \gamma^3+\Theta_1\gamma^2\lambda+\Theta_2\gamma\lambda^2+ | b_{ij} | \lambda^3,$$

where $$\Theta_1 = \sum A_{ij}b_{ij}, \quad \Theta_2 = \sum B_{ij}a_{ij}.$$

Apolar conics. Assume that $\phi_1 = 0$ and $\phi_2 = 0$ are nondegenerate conics and let us, without loss of generality, make a special choice of the coordinate system. Let the triangle of reference $A_1A_2A_3$ be so chosen that it is self-polar with respect to $\phi_1 = 0$ and that its two vertices A_1, A_2 lie on $\phi_2 = 0$. Then

$$\phi_1 \equiv \sum a_{ii}x_i^2 \quad \text{and} \quad b_{11} = b_{22} = 0$$

So, $$\Theta_1 = A_{33}b_{33}$$

Since $A_{33} \neq 0$, $\Theta_1 = 0$ if and only if $b_{33} = 0$, that is, when and only when A_3 lies on $\phi_2 = 0$. We have thus the following properties :

The necessary and sufficient condition that there exists a triangle which is self-polar with respect to $\phi_1 = 0$ and whose vertices lie on $\phi_2 = 0$ is $\Theta_1 = 0$. Similarly for $\Theta_2 = 0$. Again, since Θ_1 and Θ_2 are dual, we have the following : The necessary and sufficient condition that there exists a triangle which is self-polar with respect to $\psi_1 = 0$ (*i.e*, $\phi_1 = 0$) and whose sides are lines of envelope of $\psi_2 = 0$ (*i.e*, touch $\phi_2 = 0$) is $\Theta_2 = 0$. Similarly for $\Theta_1 = 0$. We have therefore the following theorem :

The necessary and sufficient condition that there exists a triangle which is self-polar with respect to one conic and inscribed in a second conic is that there exists a triangle which is self-polar with respect to the second conic and circumscribed about the first.

Two conics which are so related are said to be *apolar*. One or both of two apolar conics may be degenerate. The relation between two apolar conics remains unaltered by collineation.

Harmonic conics. As before, let $\phi_1 = 0$ and $\phi_2 = 0$ be two conics. Take as the triangle of reference the triangle which is self-polar with respect to both the conics. Then the equations of the conics can be reduced to the forms :

$$\phi_1 \equiv ax_1^2 + bx_2^2 + cx_3^2 = 0, \quad \phi_2 \equiv x_1^2 + x_2^2 + x_3^2 = 0$$

And so, in line coordinates, the first conic has the equation

$$\psi_1 \equiv bcu_1^2 + cau_2^2 + abu_3^2 = 0$$

Let us find the locus of a point $P = (x_1', x_2', x_3')$ such that the tangents from P to one of the conics $\phi_1 = 0$, $\phi_2 = 0$ are harmonically separated by the tangents from P to the other conic The condition is the same as that the tangents from P to the conic $\phi_2 = 0$ is apolar to the conic $\psi_1 = 0$. Now, the equation of the pair of tangents to $\phi_2 = 0$ is

$$(x_1^2 + x_2^2 + x_3^2)\,(x_1'^2 + x_2'^2 + x_3'^2) - (x_1x_1' + x_2x_2' + x_3x_3')^2 = 0$$

or
$$x_1^2(x_2'^2 + x_3'^2) + x_2^2(x_3'^2 + x_1'^2) + x_3^2(x_1'^2 + x_2'^2)$$
$$-2(x_1x_2x_1'x_2' + x_1x_3x_1'x_3' + x_2x_3x_2'x_3') = 0$$

The condition that this degenerate conic should be apolar to $\psi_1 = 0$ is

$$bc(x_2'^2 + x_3'^2) + ca(x_3'^2 + x_1'^2) + ab(x_1'^2 + x_2'^2) = 0$$

Hence the locus of P is

$$F \equiv a(b+c)x_1^2 + b(c+a)x_2^2 + c(a+b)x_3^2 = 0 \qquad (11.22)$$

The locus is therefore a conic ; it is called the *harmonic conic locus* of $\phi_1 = 0$ and $\phi_2 = 0$.

Similarly, it can be shown that the envelope of a line which is cut harmonically by the two conics $\phi_1 = 0$, $\phi_2 = 0$ is a conic envelope whose equation is

$$\Phi \equiv (b+c)u_1^2 + (c+a)u_2^2 + (a+b)u_3^2 = 0 \qquad (11.23)$$

This conic is called the *harmonic conic envelope* of $\phi_1 = 0$ and $\phi_2 = 0$.

It may be seen that the points of contact of common tangents to $\phi_1 = 0$ and $\phi_2 = 0$ lie on (11.22) and that the tangents to $\phi_1 = 0$ and $\phi_2 = 0$ at their common points are tangents to (11.23). Also, these four conics have a common self-polar triangle. If therefore we apply a collineation of the plane, then the conics into which F-conic and Φ-conic are transformed will remain respectively the harmonic conic locus and the harmonic conic envelope of the two conics into which $\phi_1 = 0$ and $\phi_2 = 0$ are transformed.

43. Affine transformations in projective geometry. Consider a collineation

$$\rho x_i' = \sum_k a_{ik} x_k, \quad i = 1, 2, 3, \quad | a_{ik} | \neq 0$$

and let us suppose that this collineation leaves a particular line, say $x_3 = 0$, fixed. Then

$$a_{31} = a_{32} = 0, \qquad | a_{ik} | = (a_{11}a_{22} - a_{12}a_{21})a_{33}$$

Hence $a_{33} \neq 0$ and we may assume, without loss of generality, that $a_{33} = 1$. The collineation therefore takes the form

$$\begin{aligned} \rho x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \rho x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ \rho x_3' &= \qquad\qquad\qquad\quad\; x_3 \end{aligned} \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \qquad (11.24)$$

In nonhomogeneous coordinates, this transformation may be written as

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13} \\ y' &= a_{21}x + a_{22}y + a_{23} \end{aligned}$$

These equations describe the transformation of all points which are not situated on the line $x_3 = 0$ and it has the form of an affinity (7.1). Hence the collineations (11.24) of the projective plane, for which a particular line remains invariant, are in a one-to-one correspondence with the affinities of the Euclidean plane and constitute the affine transformations of the projective plane. We can therefore investigate *affine geometry in the projective plane*. For this purpose we distinguish one particular line and consider only those collineations for which this particular line remains invariant. This invariant line is usually called the *line at infinity*.

Two figures have been called projective (affine) if there exists a collineation (an affine transformation) which carries one figure into the other ; *e.g.*, two real nondegenerate conics are projective. It follows that two affine figures are projective, but two projective figures are not necessarily affine ; *e.g.*, two projective figures, one of which meets the line at infinity while the other does not, cannot be affine. In particular, a pair of distinct lines is projective to every figure of the same type, but no two of the pairs of lines

$$x_1 x_2 = 0, \quad x_1(x_1 + x_3) = 0, \quad x_1 x_3 = 0$$

($x_3 = 0$ being the line at infinity) are affine.

Two affine conics are projective and the matrices of the corresponding polarities must therefore be of equal rank. We shall consider here only the cases where the rank is *three* and see how the two classes of projective conics are split up into classes of affine conics.

Let the equation of a conic be

$$\sum c_{ij}x_ix_j = 0, \quad c_{ij} = c_{ji},$$

and that of the line at infinity be $x_3 = 0$. Two cases arise according as the line at infinity does not or does pass through its pole with respect to the polarity generated by the conic.

(1) When the line at infinity does not pass through its pole, there exists a polar triangle $A_1A_2A_3$, where the points A_1, A_2 are situated on the line at infinity. If this polar triangle be taken as the triangle of reference, the equation of the conic is transformed as

$$c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 = 0, \quad c_{11}c_{22}c_{33} \neq 0$$

Putting $$c_{11} = \pm a^2, \quad c_{22} = \pm b^2, \quad c_{33} = \pm c^2$$

and applying the affine transformation

$$ax_1 = x'_1, \quad bx_2 = x'_2, \quad cx_3 = x'_3$$

and finally dropping the dashes, we obtain for the conic the equation

$$\pm x_1^2 \pm x_2^2 = x_3^2 \tag{11.25}$$

We may interchange x_1 and x_2 but not x_1 (or x_2) and x_3, as $x_3 = 0$ should remain invariant. It follows from (11.25) that the given conic is affine to one of the following conics :

Projective forms	Intersections with the line at infinity	Nonhomogeneous forms	Types
$x_1^2 + x_2^2 = x_3^2$	nil	$x^2 + y^2 = 1$	ellipse (circle)
$x_1^2 - x_2^2 = x_3^2$	(1, 1, 0), (1, −1, 0)	$x^2 - y^2 = 1$	hyperbola
$-x_1^2 - x_2^2 = x_3^2$	nil	$-x^2 - y^2 = 1$	no real trace

Obviously no two of these three types of conics are affine.

(2) When the line at infinity passes through its pole, let the coordinates be so chosen that the pole A_1 of the line at infinity $x_3 = 0$ has the coordinates (0, 1, 0). As this pole is a real point of the nucleus, the line $x_1 = 0$, which passes through this point, must intersect the conic in a second point A_3 which we choose to be (0, 0, 1). Let the coordinates

of the pole A_2 of $x_1 = 0$, which lies on $x_3 = 0$, be chosen as (1, 0, 0). Then

$$c_{12} = c_{21} = c_{13} = c_{31} = c_{22} = c_{33} = 0$$

Furthermore, let A be any point on the line at infinity ; its polar passes through A_1 and intersects the line AA_3 in a point B which we choose as the unit point (1, 1, 1). Thus A is represented by (1, 1, 0) and its polar A_1B by (1, 0, −1). So

$$c_{23} = c_{32} = -c_{11}$$

The equation of the conic therefore reduces to the form

$$x_1^2 - 2x_2x_3 = 0 \qquad (11.26)$$

Thus the given conic is affine to the following conic :

Projective form	Intersection with the line at infinity	Nonhomogeneous form	Type
$x_1^2 = 2x_2x_3$	tangent	$x^2 = 2y$	parabola

From these considerations it follows that

(*a*) Nondegenerate conics without real trace are affine.

(*b*) Real nondegenerate conics (which are projective) can be divided into three types of conics : ellipses (not intersecting the line at infinity), hyperbolas (intersecting the line at infinity in two points) and parabolas (osculating the line at infinity). Conics of the same type are affine.

It is seen from above that the affine classification of conics depends on their relations with the line at infinity. We have seen in § 29.1 that the middle point of a line segment is the harmonic conjugate of the point at infinity of the line with respect to the two extremities of the segment. It follows that *the polar of the centre of a central conic with respect to the conic is the line at infinity.* So, when the line at infinity does not meet the conic, the centre is inside the conic and the conic is an ellipse. When the line at infinity meets the conic in two distinct points, the centre is outside the conic and the conic is a hyperbola ; there must therefore exist two tangents to a hyperbola drawn from its centre ; they are the asymptotes which touch the curve at the points where it is intersected by the line at infinity. Finally, the line at infinity touches a parabola at the point where it is intersected by the axis of the curve.

Two conjugate diameters of a central conic have been defined in § 24.1 as two diameters each of which contains the middle points of a system of chords parallel to the other. Let a, b be two conjugate diameters. All lines parallel to b meet in the point at infinity of b. So, the pole of a is the point at infinity of b and the pole of b is the point at infinity of a. Therefore two conjugate diameters are conjugate in the sense that each passes through the pole of the other. Hence, it follows from § 27.1 that the pairs of conjugate diameters form an involution of lines. This involution is hyperbolic in the case of a hyperbola, the asymptotes being the double lines ; and the involution is elliptic in the case of an ellipse. In the case of a parabola, this involution does not exist. A diameter of a parabola (a line parallel to the axis) and a line parallel to a chord bisected by the diameter are conjugate lines.

44. Metric properties in the projective plane. Consider a polarity transforming points into lines and its dual transforming lines into points. It has been seen in § 39 that, if the rank of the matrix of the polarity is three, the two polarities are not distinct.

Consider the case where the rank is *two*. Obviously the transformation of a matrix to a normal form does not depend upon the variables which are to be transformed by the matrix ; they may be denoted by (x_1, x_2, x_3) to represent points or by (u_1, u_2, u_3) to represent lines. We take the case of the polarity as a line-to-point transformation for which, by (11.7) and (11.8), there are two normal forms. We have for these forms the following scheme which is explained below it

$\rho x_1 = u_1,\ \rho x_2 = u_2,\ \rho x_3 = 0$	$\rho x_1 = u_1,\ \rho x_2 = -u_2,\ \rho x_3 = 0$
$u_1v_1 + u_2v_2 = 0$	$u_1v_1 - u_2v_2 = 0$
$u_1^2 + u_2^2 = 0$	$u_1^2 - u_2^2 = 0$
$x_1^2 + x_2^2 = 0, \quad x_3 = 0$	$x_1^2 - x_2^2 = 0, \quad x_3 = 0$
$(1, i, 0), \ (1, -i, 0)$	$(1, 1, 0), \ (1, -1, 0)$

The first line gives, in normal forms, the two polarities ; in both cases all the poles are situated on the line $x_3 = 0$. The second line gives the condition that the line (v_1, v_2, v_3) passes through the pole of (u_1, u_2, u_3). The third line contains the equations of the lines conjoint with their poles; they form a pair of pencils. The equations of the centres of these pencils are given in the next line and the coordinates of the centres are contained in the last line.

We shall consider the scheme on the *left side only*. The equation $u_1v_1 + u_2v_2 = 0$ can be interpreted in Euclidean plane geometry and in (orthogonal) Cartesian coordinates as the condition of orthogonality of the two lines

$$u_1x + u_2y + u_3 = 0, \quad v_1x + v_2y + v_3 = 0$$

Now take a collineation. By (11.3), (11.3″), the collineation and its dual can be expressed as

$$\rho x_i' = \sum a_{ik} x_k, \quad \sigma u_i = \sum a_{ki} u'_k, \quad i = 1, 2, 3, \quad | a_{ik} | \neq 0$$

Let the line $x_3 = 0$ be left fixed by the collineation and let this line be regarded as the line at infinity. So

$$a_{31} = a_{32} = 0$$

Further suppose that this collineation preserves the polarity which we are considering (on the left side of the scheme). Hence orthogonality is preserved. So, $u_1v_1 + u_2v_2 = 0$ is transformed into $u_1'v_1' + u_2'v_2' = 0$. Comparing coefficients,

$$a_{11}^2 + a_{12}^2 = a_{21}^2 + a_{22}^2, \quad a_{11}a_{21} + a_{12}a_{22} = 0$$

Therefore we may put

$$\begin{aligned} a_{11} &= c \cos\theta, & a_{12} &= c \sin\theta \\ a_{21} &= \mp c \sin\theta, & a_{22} &= \pm c \cos\theta \end{aligned}$$

Thus the collineation reduces to the form

$$\begin{aligned} \rho x_1' &= c \cos\theta \, x_1 + c \sin\theta \, x_2 + c_1 x_3 \\ \rho x_2' &= \mp c \sin\theta \, x_1 \pm c \cos\theta \, x_2 + c_2 x_3 \\ \rho x_3' &= x_3 \end{aligned} \tag{11.27}$$

It can be so arranged that outside the line at infinity the coordinates $x = x_1/x_3$, $y = x_2/x_3$ are the (orthogonal) Cartesian coordinates. Hence, in nonhomogeneous coordinates, the transformation (11.27) is a similarity transformation (of the form (5.6))

$$\begin{aligned} x' &= c\,(\cos\theta \, x + \sin\theta \, y + d_1) \\ y' &= \pm c\,(-\sin\theta \, x + \cos\theta \, y + d_2) \end{aligned}$$

of the Euclidean plane. Using complex coordinates (§ 14.1) we may arrive at the same result also in the following manner:

The collineations for which the polar field in question remains invariant preserve the line $x_3 = 0$ on which every pole is situated. Hence these collineations are affinities. Further, since the polar field is invariant, the points $x_1^2 + x_2^2 = 0$, $x_3 = 0$ remain fixed. Hence every isotropic line is transformed into an isotropic line. The angle between two (real) lines p_1, p_2 has been represented, by (4.21), as $\log (p_1 p_2, p_I p_J)/2i$ where p_I, p_J are the isotropic lines passing through the intersection of p_1, p_2. Hence, if the two kinds of isotropic lines (or alternatively, the two points $(1, i, 0)$, $(1, -i, 0)$) are interchanged, then either the angle remains invariant or it changes its sign. The transformations (*i.e.*, the affinities) are therefore the similarities.

Every conic can be represented by

$$q(x_1, x_2) + x_3\, l(x_1, x_2, x_3) = 0$$

where $$q(x_1, x_2) = c_{11}x_1^2 + 2c_{12}x_1x_2 + c_{22}x_2^2$$

and $$l(x_1, x_2, x_3) = 2c_{13}x_1 + 2c_{23}x_2 + c_{33}x_3$$

are a quadratic and a linear form respectively. Two points $(a_1, a_2, 0)$ and $(b_1, b_2, 0)$ of the line $x_3 = 0$ are conjugate with respect to the polarity generated by the conic if

$$c_{11}a_1b_1 + c_{12}(a_1b_2 + a_2b_1) + c_{22}a_2b_2 = 0$$

The involution generated by the polarity on the line $x_3 = 0$ therefore depends only on the quadratic form $q(x_1, x_2)$. On the other hand, the quadratic form is determined, up to a factor $\neq 0$, by that polarity. It therefore follows that the conics, which generate on the line at infinity $x_3 = 0$ the same involution as does the polarity which we are considering in the above scheme, are those that are given by the equations

$$x_1^2 + x_2^2 + x_3\, l(x_1, x_2, x_3) = 0,$$

where l is an arbitrary linear form. Going back to the system of non-homogeneous coordinates, it is seen that these conics are circles so far as they have real traces. Thus, *all circles generate the same involution on the line at infinity.* This result has already been obtained before in § 42.

The points of intersection of a circle and the line at infinity are given by the last two lines of the scheme, namely

$$x_1^2 + x_2^2 = 0, x_3 = 0, \quad i.e., \quad (1, i, 0), (1, -i, 0) \qquad (11.28)$$

It follows that *all circles pass through the same two (conjugate imaginary) poins (11.28) of the line at infinity.* These points are therefore called *the circular points at infinity.* On the other hand, *every conic which passes through the two circular points at infinity is a circle.* For, the points of intersection of a conic $\sum c_{ij} x_i x_j$ with the line at infinity are given by

$$c_{11}x_1^2 + 2c_{12}x_1x_2 + c_{22}x_2^2 = 0, \quad x_3 = 0$$

If these points are the circular points,

$$c_{11} = c_{22}, \quad c_{12} = 0$$

Therefore the conic is a circle. Also, it is evident that if a conic passes through one of the circular points, it passes through the other.

If the equations of rigid motions (3.1) are written in homogeneous coordinates, then, for these transformations, the circular points will remain fixed. *The metric properties of conics in the projective plane* have therefore to do with the two circular points at infinity which are denoted by the letters I, J. The two isotropic lines through an ordinary point P are the lines joining P to I and J.

The equations of the tangents to a circle

$$x_1^2 + x_2^2 + 2c_1x_1x_3 + 2c_2x_2x_3 + c_3x_3^2 = 0$$

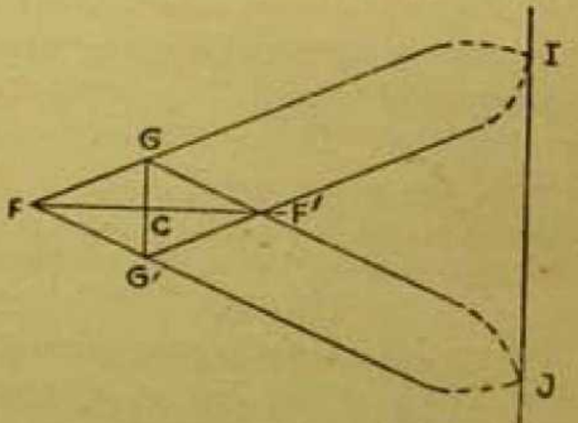
at the points I and J are

$$(x_1 + c_1x_3) + i(x_2 + c_2x_3) = 0, \quad (x_1 + c_1x_3) - i(x_2 + c_2x_3) = 0$$

These tangents evidently pass through the centre of the circle. The isotropic tangents to a circle are therefore sometimes called the *asymptotes of the circle.*

The axis and the tangent at the vertex of a parabola are orthogonal. So, they are harmonically separated by the two isotropic lines through the vertex of the parabola. Hence the pole of the axis of a parabola is the harmonic conjugate of the point at infinity of the parabola with respect to I and J. Also, the lines joining I and J to the focus of a parabola are the isotropic tangents to the parabola.

We have seen in § 14.1 that there are four isotropic tangents to a central conic and any two nonparallel isotropic tangents intersect in a focus, there being four foci, two real and two conjugate imaginaries. Let F, F' be the real foci and G, G' be the conjugate imaginary foci of a central conic and let I and J be the circular points. Then the parallel isotropic tangents FG, $F'G'$ intersect in a circular point I, the two other parallel isotropic tangents FG', $F'G$ intersect in the other circular point J. The two lines FG, FG', as also the two lines $F'G$, $F'G'$, are conjugate imaginary lines. The lines FF' and GG' are the axes and their point of intersection, C, is the centre of the conic.

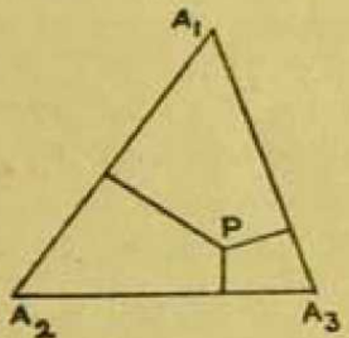


APPENDIX I

TRILINEAR AND AREAL COORDINATES

Consider an arbitrary but fixed triangle whose vertices are A_1, A_2, A_3 and whose sides opposite to them are, as usual, a_1, a_2, a_3; the angles and the lengths of the sides of this triangle shall also be denoted by these same letters, as is usually done. Let the area of the triangle for a definite cyclic permutations of the vertices be denoted by Δ and, without loss of generality, we shall suppose that Δ is a positive quantity. Thus, the area is positive for a definite sense (say, counter-clockwise) of going round the triangle and negative for the opposite sense.

Let P be a point in the plane and let the perpendicular distances of P from the sides a_1, a_2, a_3 be denoted by $\omega_1, \omega_2, \omega_3$ respectively. Then the perpendicular ω_i shall be considered positive or negative according as the sense of going round the triangle PA_jA_k is the same or the opposite as that of going round the triangle $A_1A_2A_3$, where *ijk* is an even permutation of 123; for example, ω_1 is positive or negative according as P and A_1 are on the same or opposite sides of a_1. The quantities $(\omega_1, \omega_2, \omega_3)$ thus introduced are called the *trilinear coordinates* of P with reference to the triangle $A_1A_2A_3$. Every point P of the plane has a unique ordered triplet as its trilinear coordinates.



Whatever be the position of P, we have the identical relation

$$a_1\omega_1 + a_2\omega_2 + a_3\omega_3 = 2\Delta \tag{1}$$

Therefore, an ordered triplet satisfying the condition (1) represents a point uniquely. For the sake of simplicity, let the origin of the rectangular Cartesian coordinates be taken *within* the triangle $A_1A_2A_3$. Let three unit vectors whose initial points are fixed at the origin and which are perpendicular to the sides a_1, a_2, a_3 with directions away from these sides be respectively $(\cos\theta_1, \sin\theta_1)$, $(\cos\theta_2, \sin\theta_2)$, $(\cos\theta_3, \sin\theta_3)$ and let the equations, in Hessian normal forms, of a_1, a_2, a_3 be

$$x\cos\theta_1 + y\sin\theta_1 + p_1 = 0, \quad x\cos\theta_2 + y\sin\theta_2 + p_2 = 0,$$
$$x\cos\theta_3 + y\sin\theta_3 + p_3 = 0$$

Then we have

$$\begin{aligned} \omega_1 &= x\cos\theta_1 + y\sin\theta_1 + p_1 \\ \omega_2 &= x\cos\theta_2 + y\sin\theta_2 + p_2 \\ \omega_3 &= x\cos\theta_3 + y\sin\theta_3 + p_3 \end{aligned} \qquad (2)$$

It is evident from the linear relations (2) that if an equation in trilinear coordinates be transformed into an equation in Cartesian coordinates, the degree of the equation will remain unchanged. Moreover, it follows from (1) and (2) that any equation in trilinear coordinates can be made *homogeneous* by multiplying each term by the factor $(a_1\omega_1 + a_2\omega_2 + a_3\omega_3)/2\Delta$ raised to suitable power. Thus the equations

$$\sum_i c_i\omega_i = 0, \quad \sum_{i,j} a_{ij}\omega_i\omega_j = 0$$

represent, in trilinear coordinates, a line and a conic respectively. The equations of the tangent and the polar of a point are therefore of the same form as given by (9.18).

Now, when we are dealing with homogeneous equations, we do not require the actual coordinates $\omega_1, \omega_2, \omega_3$, but we need only the ratios $\omega_1 : \omega_2 : \omega_3$. Hence any three quantities which are proportional to $\omega_1, \omega_2, \omega_3$ may be taken as the coordinates of the point. Thus the coordinates of A_1, A_2, A_3 may be taken as $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively. It may be verified that the coordinates of the following points with respect to the triangle of reference may be chosen as follows :

In-centre : $(1, 1, 1)$; ex-centre opposite A_1 : $(-1, 1, 1)$;

circumcentre : $(\cos A_1, \cos A_2, \cos A_3)$;

centroid : $(\sin A_2 \sin A_3, \sin A_3 \sin A_1, \sin A_1 \sin A_2)$;

orthocentre : $(\cos A_2 \cos A_3, \cos A_3 \cos A_1, \cos A_1 \cos A_2)$;

nine-point centre : $\big(\cos(A_2 - A_3),\ \cos(A_3 - A_1),\ \cos(A_1 - A_2)\big)$.

The equations of the sides a_1, a_2, a_3 are $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = 0$ respectively and of the medians of the triangle of reference are

$$\omega_2 \sin A_2 = \omega_3 \sin A_3, \quad \omega_3 \sin A_3 = \omega_1 \sin A_1, \quad \omega_1 \sin A_1 = \omega_2 \sin A_2$$

The points for which the expression on the left-hand side of (1) vanishes cannot be the ordinary points ; they are taken as the *points at infinity*. So, the equation of the *line at infinity* is

$$a_1\omega_1 + a_2\omega_2 + a_3\omega_3 = 0 \qquad (3)$$

Accordingly, if the two lines $b_1\omega_1 + b_2\omega_2 + b_3\omega_3 = 0$, $c_1\omega_1 + c_2\omega_2 + c_3\omega_3 = 0$ are parallel, they must meet the line (3) in the same point. The condition for this is $|\, a\, b\, c \,| = 0$. If from any point $(\omega_1, \omega_2, \omega_3)$ of the circum-circle of the triangle of reference perpendiculars are drawn on the sides of the triangle, the feet of the perpendiculars are collinear.

Therefore
$$\omega_2\omega_3 \sin A_1 + \omega_3\omega_1 \sin A_2 + \omega_1\omega_2 \sin A_3 = 0$$
or
$$a_1\omega_2\omega_3 + a_2\omega_3\omega_1 + a_3\omega_1\omega_2 = 0 \tag{4}$$

This, therefore, is the equation of the circum-circle of the triangle of reference. Hence the equation of any circle can be written as

$$a_1\omega_2\omega_3 + a_2\omega_3\omega_1 + a_3\omega_1\omega_2 + (c_1\omega_1 + c_2\omega_2 + c_3\omega_3)(a_1\omega_1 + a_2\omega_2 + a_3\omega_3) = 0, \tag{5}$$

where c_1, c_2, c_3, are arbitrary constants.

Application. It may be verified that if the circle (5) passes through the feet of the perpendiculars drawn from the vertices to the opposite sides of the triangle of reference, then

$$2c_1 = \cos A_1, \quad 2c_2 = \cos A_2, \quad 2c_3 = \cos A_3$$

Hence the equation of the nine-point circle of the triangle of reference is

$$2(a_1\omega_2\omega_3 + a_2\omega_3\omega_1 + a_3\omega_1\omega_2) \\ + (\omega_1 \cos A_1 + \omega_2 \cos A_2 + \omega_3 \cos A_3)(a_1\omega_1 + a_2\omega_2 + a_3\omega_3) = 0 \tag{6}$$

Again, if the vertices $A_1 = (1, 0, 0)$, $A_2 = (0, 1, 0)$, $A_3 = (0, 0, 1)$ are the poles, with respect to the circle (5), of the opposite sides $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = 0$ respectively, it may be verified that

$$c_1 = \cos A_1, \quad c_2 = \cos A_2, \quad c_3 = \cos A_3$$

Therefore the equation of the circle with respect to which the triangle of reference is a self-polar triangle is

$$a_1\omega_2\omega_3 + a_2\omega_3\omega_1 + a_3\omega_1\omega_2 \\ + (\omega_1 \cos A_1 + \omega_2 \cos A_2 + \omega_3 \cos A_3)(a_1\omega_1 + a_2\omega_2 + a_3\omega_3) = 0 \tag{7}$$

Hence, it follows from the equations (4), (6), (7) that *the circum-circle, the nine-point circle and the self-conjugate circle of a triangle are coaxial.*

Another system of coordinates which is often used is as follows : Take the triangle of reference $A_1A_2A_3$ with the two senses, positive and negative, of going round the triangle as explained before. Let P be any point of the plane and let the areas (with proper signs) of the triangles PA_2A_3, PA_3A_1, PA_1A_2 be denoted by Δ_1, Δ_2, Δ_3 respectively. If three quantities z_1, z_2, z_3 are introduced such that

$$\frac{z_1}{\Delta_1} = \frac{z_2}{\Delta_2} = \frac{z_3}{\Delta_3} = \frac{1}{\Delta},$$

then (z_1, z_2, z_3), are called the *areal coordinates* of P.

If $(\omega_1, \omega_2, \omega_3)$ are the trilinear coordinates of P, then the relation between the trilinear and areal coordinates is given by

$$z_1 = \frac{a_1\omega_1}{2\Delta}, \quad z_2 = \frac{a_2\omega_2}{2\Delta}, \quad z_3 = \frac{a_3\omega_3}{2\Delta}$$

Hence, it follows from (1) that the areal coordinates are connected by the identical relation

$$z_1 + z_2 + z_3 = 1 \tag{8}$$

The equation of the line at infinity in areal coordinates is

$$z_1 + z_2 + z_3 = 0 \tag{9}$$

Also, as in the case of trilinear coordinates, it is seen that with areal coordinates our equations can also be made homogeneous.

SPACE GEOMETRY

CHAPTER XII

THE EUCLIDEAN SPACE

45. Points and vectors. In space, there exists one and only one straight line passing through two distinct points and one and only one plane through three distinct noncollinear points or through two distinct intersecting straight lines or through a straight line and a point outside it or through two parallel straight lines. Through a point outside a given plane, there exists one and only one plane parallel to the given plane; and any straight line drawn on one of two parallel planes is parallel to the other.

Lemma I. All straight lines perpendicular to a given straight line at a given point of it form a pencil of lines and lie in a plane.

The given straight line and the plane are said to be perpendicular or normal to one another.

Lemma II. Through every point of space there exists one and only one straight line normal to a given plane.

Any plane passing through a normal to a plane is said to be normal to the given plane.

By using the above two lemmas we can, through every point of space, construct, in an infinite number of ways, three and only three straight lines which are mutually orthogonal. The three planes passing through the three pairs of three such mutually orthogonal straight lines are mutually orthogonal and they divide the space into eight regions.

Let us take three fixed mutually orthogonal straight lines, called the x-, the y- and the z- *axes of coordinates*, through a point O, called the *origin*. The origin divides each of the axes into two half-rays. We then choose, in an arbitrary but fixed manner, the positive (and therefore also the negative) half-rays of the axes. Having thus chosen the positive and the negative directions along the axes, let us view the (y, z)- plane from a point of the positive x-axis and imagine the positive half-ray of the y-axis to rotate about the origin in the (y, z)- plane in the positive (*i.e.*, the counter-clockwise) sense through an angle $\pi/2$. We then say that we have a *right-handed* system of axes if, after rotation, the positive half-rays of the y- and the z-axes coincide, otherwise the system is *left-handed.* When the system is right-handed, the positive z-axis, after rotating about the origin through $+\pi/2$ in the (z, x)- plane coincides with the positive x-axis when viewed from a point of the positive y-axis; and similarly, the positive x-axis coincides with the positive y-axis when viewed from a point of the positive z-axis. We shall suppose that our coordinate system is right-handed.

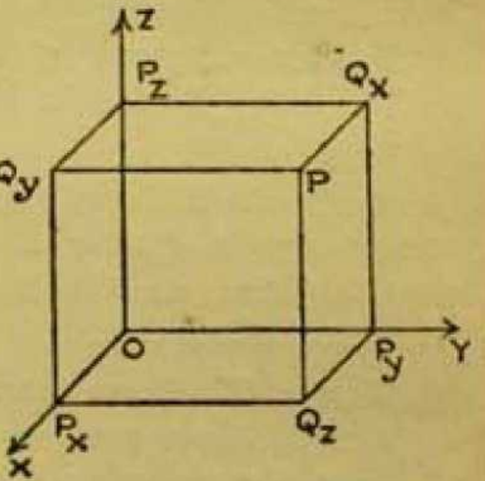
If now we choose three congruent unit segments on the three axes, it is seen, as in Chap. I, that to every point of an axis there corresponds a real number (the coordinate) and, conversely, to every real number there corresponds one point of each axis. Let P be any point of space and through P let three planes parallel to the (y, z)-, (z, x)- and (x, y)- planes be drawn to meet the x-, y- and z- axes in the points P_x, P_y, P_z respectively. Then the position of the point P is uniquely determined by the coordinates x of P_x, y of P_y, z of P_z of the axes. Thus, to every point P of space there corresponds a triplet of numbers (x, y, z), called its *coordinates*, and conversely. To denote a point by its coordinates, we shall write

$$P = (x, y, z)$$

The distance between the two points

$$P = (x, y, z) \text{ and } O = (0, 0, 0)$$

is given by



$$|OP| = |\sqrt{(x^2+y^2+z^2)}| \tag{12.1}$$

As in the plane geometry, we introduce *directed segments* and *vectors* in space. If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are two points, then the coordinates of the directed segment $\overline{P_1P_2}$ will be given by the coordinates of the three directed segments which are the orthogonal projections of $\overline{P_1P_2}$ on the three axes, *i.e.*, they are given by $(x_2-x_1, \ y_2-y_1, \ z_2-z_1)$; and so its length is, by (12.1), given by the distance

$$|P_1P_2| = |\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\}}|$$

Directed segments with the same coordinates represent the same *vector* and a vector has the same coordinates and the same length as those of any one of the directed segments representing it. Thus, we may write, as vectors,

$$\overline{P_1P_2} = (x_2-x_1, \ y_2-y_1, \ z_2-z_1) \tag{12.2}$$

If we draw through P the straight line parallel to the z-axis to meet the (x, y)- plane in Q_z and complete the rectangles $PQ_zP_yQ_x$, $PQ_xP_zQ_y$, we have, as vectors,

$$\overline{OP_x} = \overline{P_yQ_z} = \overline{Q_xP} = \overline{P_zQ_y}\,;$$

$$\overline{OP_y} = \overline{P_zQ_x} = \overline{Q_yP} = \overline{P_xQ_z}\,;$$

$$\overline{OP_z} = \overline{P_xQ_y} = \overline{Q_zP} = \overline{P_yQ_x}\,;$$

Let $P_1, P_2, \ldots, P_n$ be any n points in space. We *define* addition of vectors by

$$\overline{P_1P_2} + \overline{P_2P_3} + \ldots + \overline{P_{n-1}\,P_n} = \overline{P_1P_n}. \tag{12.3}$$

If $\overline{P_iP_{i+1}} = (\xi_i, \eta_i, \zeta_i), \qquad i = 1, 2, \ldots n-1,$

then

$$\overline{P_1P_n} = \left(\sum_{i=1}^{n-1}\xi_i,\ \sum_{i=1}^{n-1}\eta_i,\ \sum_{i=1}^{n-1}\zeta_i\right)$$

Hence the addition of two vectors is commutative and obeys the *parallelogram law.*

It follows that if $P_1 = (x_1, y_1, z_1)$, then

$$P_n = (x_1 + \sum\xi_i,\quad y_1 + \sum\eta_i,\quad z_1 + \sum\zeta_i)$$

Let a vector v have coordinates (ξ, η, ζ). Then

$$|\,v\,|^2 = \xi^2 + \eta^2 + \zeta^2,$$

and $\qquad \xi = |\,v\,| \cos(x, v), \quad \eta = |\,v\,| \cos(y, v), \quad \zeta = |\,v\,| \cos(z, v),$

where $|\,v\,|$ is the length of the vector and (x, v), (y, v) (z, v) are the angles between the positive axes of coordinates and the vector v. Squaring and adding the last three relations,

$$|\,v\,|^2 = |\,v\,|^2 \{\cos^2(x, v) + \cos^2(y, v) + \cos^2(z, v)\}$$

Therefore

$$\cos^2(x, v) + \cos^2(y, v) + \cos^2(z, v) = 1 \tag{12.4}$$

The angles (x, v), (y, v), (z, v) are called the direction-angles and cosines of these angles the *direction-cosines of the vector v.*

46. Scalar product of two vectors. Area of a triangle. Let P_1, P_2, P_3 be three points and let the following vectors have the coordinates

$$\overline{P_3P_1} = (\xi_1, \eta_1, \zeta_1), \quad \overline{P_3P_2} = (\xi_2, \eta_2, \zeta_2).$$

So $\quad |\,P_1P_2\,|^2 = (\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2 + (\zeta_2 - \zeta_1)^2$

$$= (\xi_1^2 + \eta_1^2 + \zeta_1^2) + (\xi_2^2 + \eta_2^2 + \zeta_2^2) - 2(\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2)$$

But $\qquad |\,P_3P_1\,|^2 = \xi_1^2 + \eta_1^2 + \zeta_1^2, \quad |\,P_3P_2\,|^2 = \xi_2^2 + \eta_2^2 + \zeta_2^2$

and $\qquad |\,P_1P_2\,|^2 = |\,P_3P_1\,|^2 + |\,P_3P_2\,|^2 - 2\,|\,P_3P_1\,|\;|\,P_3P_2\,| \cos\theta,$

where θ is the angle between the vectors $\overline{P_3P_1}$ and $\overline{P_3P_2}$.

Therefore $$|P_3P_1||P_3P_2|\cos\theta = \xi_1\xi_2+\eta_1\eta_2+\zeta_1\zeta_2 \tag{12.5}$$
The left-hand side of (12.5) is called the *scalar product* of the two vectors $\overline{P_3P_1}$ and $\overline{P_3P_2}$ and shall be denoted by $\overline{P_3P_1}\cdot\overline{P_3P_2}$. From (12.5) we get

$$\cos\theta = \frac{\xi_1\xi_2+\eta_1\eta_2+\zeta_1\zeta_2}{|\sqrt{\{(\xi_1^2+\eta_1^2+\zeta_1^2)(\xi_2^2+\eta_2^2+\zeta_2^2)\}}|}$$

So, $$\sin^2\theta = \frac{(\eta_1\zeta_2-\eta_2\zeta_1)^2+(\zeta_1\xi_2-\zeta_2\xi_1)^2+(\xi_1\eta_2-\xi_2\eta_1)^2}{(\xi_1^2+\eta_1^2+\zeta_1^2)(\xi_2^2+\eta_2^2+\zeta_2^2)}$$

The condition of *orthogonality* of the vectors $\overline{P_3P_1}$ and $\overline{P_3P_2}$ is therefore

$$\xi_1\xi_2+\eta_1\eta_2+\zeta_1\zeta_2 = 0$$

and that of *parallelism* is

$$(\eta_1\zeta_2-\eta_2\zeta_1)^2+(\zeta_1\xi_2-\zeta_2\xi_1)^2+(\xi_1\eta_2-\xi_2\eta_1)^2 = 0$$

Now let Δ denote, in absolute value, the area of the triangle $P_1P_2P_3$. Then

$$2\Delta = \big|\,|P_3P_1|\,|P_3P_2|\sin\theta\,\big|$$

Therefore $$4\Delta^2 = (\eta_1\zeta_2-\eta_2\zeta_1)^2+(\zeta_1\xi_2-\zeta_2\xi_1)^2+(\xi_1\eta_2-\xi_2\eta_1)^2$$

If $$P_1 = (x_1, y_1, z_1),\quad P_2 = (x_2, y_2, z_2),\quad P_3 = (x_3, y_3, z_3)$$

then, since

$$\xi_1 = x_1-x_3,\quad \eta_1 = y_1-y_3,\quad \zeta_1 = z_1-z_3,$$
$$\xi_2 = x_2-x_3,\quad \eta_2 = y_2-y_3,\quad \zeta_2 = z_2-z_3,$$

we have

$$\eta_1\zeta_2-\eta_2\zeta_1 = \begin{vmatrix} \eta_1 & \zeta_1 \\ \eta_2 & \zeta_2 \end{vmatrix} = \begin{vmatrix} y_1-y_3 & z_1-z_3 \\ y_2-y_3 & z_2-z_3 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix};$$

similarly for

$$\zeta_1\xi_2-\zeta_2\xi_1 \quad\text{and}\quad \xi_1\eta_2-\xi_2\eta_1$$

As already adopted in § 23, it will be convenient to denote

$$\begin{vmatrix} \eta_1 & \zeta_1 \\ \eta_2 & \zeta_2 \end{vmatrix} \text{ by } |\eta_1\,\zeta_2| \text{ or } |\eta\,\zeta|,\quad \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \text{ by } |y_1\,z_2\,1| \text{ or } |y\,z\,1|$$

So,
$$4\Delta^2 = |\eta\,\zeta|^2+|\zeta\,\xi|^2+|\xi\,\eta|^2 = |y\,z\,1|^2+|z\,x\,1|^2+|x\,y\,1|^2 \tag{12.6}$$
This shows incidentally that *the square of the area of a triangle is equal to the sum of the squares of its projections on the three coordinate planes.*

47. The straight line. Let (x_0, y_0, z_0) be a given point on a straight line g and (ξ, η, ζ) be a vector parallel to g. A point (x, y, z) of g is then given by

$$x = x_0 + \rho\,\xi, \quad y = y_0 + \rho\,\eta, \quad z = z_0 + \rho\,\zeta, \tag{12.7}$$

where ρ is a parameter. By giving different values to ρ we obtain different points of g and so the equations (12.7) constitute the parametric representation of g.

We can associate with every line g one of two directions along it, and speak of the line as a *directed line*. If the directed line g and the vector $u = (\xi, \eta, \zeta)$ have the same direction, the quantities

$$\frac{\xi}{|\sqrt{(\xi^2+\eta^2+\zeta^2)}|}, \quad \frac{\eta}{|\sqrt{(\xi^2+\eta^2+\zeta^2)}|}, \quad \frac{\zeta}{|\sqrt{(\xi^2+\eta^2+\zeta^2)}|},$$

which are the direction-cosines of u, are also called the *direction-cosines of the directed line* g. The direction-cosines of an oppositely directed line are obtained from the above by changing the signs of ξ, η, ζ.

Eliminating ρ, the equations (12.7) reduce to

$$(x-x_0) : (y-y_0) : (z-z_0) = \xi : \eta : \zeta$$

Hence two straight lines which are parallel to the vectors (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) are orthogonal if

$$\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2 = 0$$

and parallel if

$$\xi_1 : \eta_1 : \zeta_1 = \xi_2 : \eta_2 : \zeta_2$$

If (x_0', y_0', z_0') is a point of the straight line g,

$$(x_0'-x_0) : (y_0'-y_0) : (z_0'-z_0) = \xi : \eta : \zeta$$

Therefore, eliminating ξ, η, ζ between these equations, the equations of g can be written as

$$x - x_0 = \lambda(x_0' - x_0), \quad y - y_0 = \lambda(y_0' - y_0), \quad z - z_0 = \lambda(z_0' - z_0), \tag{12.8}$$

where λ is an arbitrary constant. Therefore the equations of g may also be written as

$$\begin{aligned} x &= \gamma x_0 + \lambda x_0' \\ y &= \gamma y_0 + \lambda y_0' \qquad \gamma + \lambda = 1 \\ z &= \gamma z_0 + \lambda z_0' \end{aligned} \tag{12.9}$$

Two straight lines in space may be either coplanar or skew. If they are coplanar, they may be either intersecting or parallel. Let the two straight lines

$$\begin{aligned} x &= x_1 + \rho\xi_1 & & & x &= x_2 + \sigma\xi_2 \\ y &= y_1 + \rho\eta_1 & &\text{and} & y &= y_2 + \sigma\eta_2 \\ z &= z_1 + \rho\zeta_1 & & & z &= z_2 + \sigma\zeta_2 \end{aligned}$$

have a point in common. Then for some particular values of ρ and σ we must have

$$\begin{aligned} x_1+\rho\xi_1 &= x_2+\sigma\xi_2 \\ y_1+\rho\eta_1 &= y_2+\sigma\eta_2 \\ z_1+\rho\zeta_1 &= z_2+\sigma\zeta_2 \end{aligned}$$

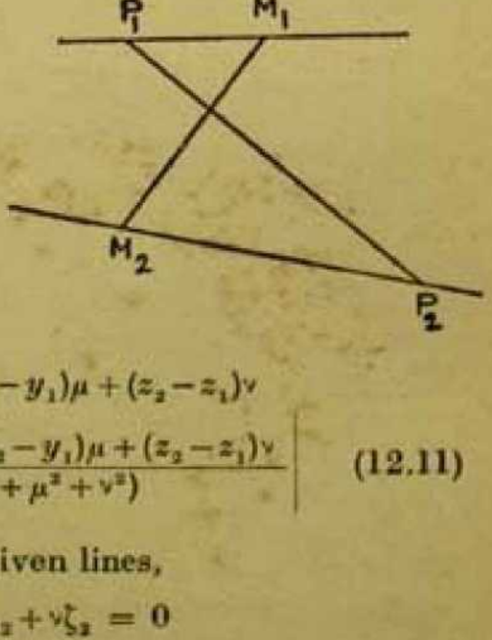
We can therefore eliminate ρ and σ between these equations and obtain the *condition of intersection* of the two straight lines as

$$\begin{vmatrix} x_1-x_2 & \xi_1 & \xi_2 \\ y_1-y_2 & \eta_1 & \eta_2 \\ z_1-z_2 & \zeta_1 & \zeta_2 \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} x_1 & x_2 & \xi_1 & \xi_2 \\ y_1 & y_2 & \eta_1 & \eta_2 \\ z_1 & z_2 & \zeta_1 & \zeta_2 \\ 1 & 1 & 0 & 0 \end{vmatrix} = 0 \qquad (12.10)$$

Shortest distance between two skew lines. Given two skew lines p_1 and p_2, there always exists a straight line which intersects the given lines orthogonally. For, through one of the given lines, say p_1, construct the plane α parallel to p_2 and through p_2 construct the plane β perpendicular to α; if β meets p_1 in M_1, then the line through M_1 which is perpendicular to α (and therefore lies in β) is the required line. If this common perpendicular meets p_2 in M_2, then $|M_1M_2|$ is the *shortest distance* between p_1 and p_2.

Let $P_1=(x_1, y_1, z_1)$ and $P_2=(x_2, y_2, z_2)$ be two points on p_1 and p_2 and let these skew lines be parallel respectively to the vectors

$$(\xi_1, \eta_1, \zeta_1), \quad (\xi_2, \eta_2, \zeta_2).$$



The shortest distance d is the orthogonal projection of the segment P_1P_2 on the line M_1M_2. If (λ, μ, ν) are the coordinates of the vector $\overline{M_1M_2}$,

$$\overline{P_1P_2} \cdot \overline{M_1M_2} = (x_2-x_1)\lambda+(y_2-y_1)\mu+(z_2-z_1)\nu$$

Therefore
$$d = |M_1M_2| = \left|\frac{(x_2-x_1)\lambda+(y_2-y_1)\mu+(z_2-z_1)\nu}{\sqrt{(\lambda^2+\mu^2+\nu^2)}}\right| \qquad (12.11)$$

But, since M_1M_2 is perpendicular to both the given lines,

$$\lambda\xi_1+\mu\eta_1+\nu\zeta_1 = 0, \quad \lambda\xi_2+\mu\eta_2+\nu\zeta_2 = 0$$

whence
$$\lambda : \mu : \nu = |\eta\,\zeta| : |\zeta\,\xi| : |\xi\,\eta|$$

Substituting these proportional quantities for λ, μ, ν in (12.11), we obtain the required expression for the shortest distance.

48. The Plane. As a plane α may uniquely be defined by any three noncollinear points lying on it, let $P_0 = (x_0, y_0, z_0)$, $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ be any three noncollinear points of α. Then, by (12.7), the points P_1, P_2 of the straight lines P_0P_1, P_0P_2 can be expressed as

$$\begin{aligned} x_1 &= x_0 + \rho'\xi_1 & \qquad x_2 &= x_0 + \sigma'\xi_2 \\ y_1 &= y_0 + \rho'\eta_1 \quad \text{and} & \qquad y_2 &= y_0 + \sigma'\eta_2 \\ z_1 &= z_0 + \rho'\zeta_1 & \qquad z_2 &= z_0 + \sigma'\zeta_2 \end{aligned}$$

By (12.9), any point $P = (x, y, z)$ of the straight line P_1P_2 is given by

$$x = \gamma x_1 + \lambda x_2, \quad y = \gamma y_1 + \lambda y_2, \quad z = \gamma z_1 + \lambda z_2, \quad \gamma + \lambda = 1$$

Therefore, eliminating $x_1, y_1, z_1, x_2, y_2, z_2$ from the above equations, we obtain the parametric equations of α as

$$\begin{aligned} x &= x_0 + \rho\xi_1 + \sigma\xi_2 \\ y &= y_0 + \rho\eta_1 + \sigma\eta_2 \\ z &= z_0 + \rho\zeta_1 + \sigma\zeta_2 \end{aligned} \tag{12.12}$$

where ρ and σ are two parameters. The above equations can be expressed directly in terms of the coordinates of P_0, P_1, P_2. Take

$$\begin{aligned} \xi_1 &= x_1 - x_0, & \eta_1 &= y_1 - y_0, & \zeta_1 &= z_1 - z_0 \\ \xi_2 &= x_2 - x_0, & \eta_2 &= y_2 - y_0, & \zeta_2 &= z_2 - z_0 \end{aligned}$$

and write μ, ν for ρ, σ respectively. Then the equations take the form

$$\begin{aligned} x &= \lambda x_0 + \mu x_1 + \nu x_2 \\ y &= \lambda y_0 + \mu y_1 + \nu y_2 \qquad \lambda + \mu + \nu = 1 \\ z &= \lambda z_0 + \mu z_1 + \nu z_2 \end{aligned} \tag{12.13}$$

The only supposition made about the three given points is that they are noncollinear; that is, the vectors (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) in (12.12) must satisfy

$$|\eta\,\zeta|^2 + |\zeta\,\xi|^2 + |\xi\,\eta|^2 \neq 0$$

It is evident that the following vectors are all parallel to the plane α:

$$(\rho\xi_1 + \sigma\xi_2, \quad \rho\eta_1 + \sigma\eta_2, \quad \rho\zeta_1 + \sigma\zeta_2)$$

Now let a vector $v = (c_1, c_2, c_3)$ be orthogonal to both the vectors (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2). Then

$$c_1\xi_1 + c_2\eta_1 + c_3\zeta_1 = 0, \quad c_1\xi_2 + c_2\eta_2 + c_3\zeta_2 = 0$$

The vector v is therefore orthogonal to all vectors parallel to the plane; for, by the above two relations, we have

$$c_1(\rho\xi_1 + \sigma\xi_2) + c_2(\rho\eta_1 + \sigma\eta_2) + c_3(\rho\zeta_1 + \sigma\zeta_2) = 0,$$

showing that v is orthogonal to the plane. Multiplying the equations (12.12) by c_1, c_2, c_3 respectively and adding,

$$c_1x+c_2y+c_3z+c_4 = 0, \tag{12.14}$$

where $$c_4 = -(c_1x_0+c_2y_0+c_3z_0)$$

Thus the coordinates of a point of the plane satisfy the linear equation (12.14). Conversely, suppose that we are given a linear equation (12.14). If (x_0', y_0', z_0') is a solution of this equation, then

$$c_1x_0'+c_2y_0'+c_3z_0'+c_4 = 0$$

Therefore $$c_1(x-x_0')+c_2(y-y_0')+c_3(z-z_0') = 0$$

This shows that (x, y, z) are the coordinates of a point of any straight line through (x_0', y_0', z_0') perpendicular to the vector v. Hence any linear equation of the form (12.14) represents a plane unless $c_1 = c_2 = c_3 = 0$. Putting

$$a = c_1/\sqrt{(c_1^2+c_2^2+c_3^2)}, \quad b = c_2/\sqrt{(c_1^2+c_2^2+c_3^2)},$$

$$c = c_3/\sqrt{(c_1^2+c_2^2+c_3^2)}, \quad d = c_4/\sqrt{(c_1^2+c_2^2+c_3^2)},$$

the equation (12.14) can be put in the *Hessian normal form* as

$$ax+by+cz+d = 0, \qquad a^2+b^2+c^2 = 1 \tag{12,15}$$

The quantities (a, b, c) are the coordinates of a unit vector normal to the plane, and so they represent the direction-cosines of this normal vector.

If the equation

$$\sigma\, ax+\sigma\, by+\sigma\, cz+\sigma\, d = 0, \quad \sigma \neq 0,$$

which represents the same plane as given by (12.15), is also given in Hessian normal form, then

$$\sigma^2(a^2+b^2+c^2) = 1. \;\; \text{So,} \quad \sigma = \pm 1$$

Hence there are two Hessian normal forms differing only in sign.

Significance of a, b, c, d and of the two Hessian normal forms. We have

$$a\xi_1+b\eta_1+c\zeta_1 = 0, \qquad a\xi_2+b\eta_2+c\zeta_2 = 0,$$

where (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) are two nonparallel vectors parallel to the plane (12.15). So,

$$a : |\,\eta\,\zeta\,| = b : |\,\zeta\,\xi\,| = c : |\,\xi\,\eta\,|$$

$$= \pm 1 : \sqrt{\{|\,\eta\,\zeta\,|^2+|\,\zeta\,\xi\,|^2+|\,\zeta\,\eta\,|^2\}}$$

Therefore $$a = \pm\,|\,\eta\,\zeta\,| : \sqrt{(|\,\eta\,\zeta\,|^2+|\,\zeta\,\xi\,|^2+|\,\xi\,\eta\,|^2)},$$

$$b = \pm\,|\,\zeta\,\xi\,| : \sqrt{(|\,\eta\,\zeta\,|^2+|\,\zeta\,\xi\,|^2+|\,\xi\,\eta\,|^2)},$$

$$c = \pm\,|\,\xi\,\eta\,| : \sqrt{(|\,\eta\,\zeta\,|^2+|\,\zeta\,\xi\,|^2+|\,\xi\,\eta\,|^2)}$$

By (12.6), the quantity $\left|\,\sqrt{(|\,\eta\,\zeta\,|^2+|\,\zeta\,\xi\,|^2+|\,\xi\,\eta\,|^2)}\,\right|$

represents twice the area of a triangle $P_0P_1P_2$, say, where $\overline{P_0P_1}$, $\overline{P_0P_2}$ represents the vectors (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) ; and the quantities $\pm | \eta \zeta |$, $\pm | \zeta \xi |$, $\pm | \xi \eta |$ represent the orthogonal projections, with a sign, of this area on the (y, z)-, (z, x)-, (x, y)- planes respectively. We may thus state the following theorem :

The quantities a, b, c, which are the cosines of the angles between the normal to a given plane and the positive axes of coordinates, are proportional to the orthogonal projections, each taken with a certain sign, of the area of an arbitrary triangle in the given plane on the three coordinate planes. The sign of a projection is to be regarded as positive or negative according as the sense of going round the projection agrees or does not agree with the positive sense of rotation in that coordinate plane with reference to the right-handed system.

Again, let $P' = (x', y', z')$ be the foot of the perpendicular drawn from the origin to the plane (12.15) and u the vector (a, b, c).

Therefore $$u.\,\overline{P'O} = -(ax' + by' + cz')$$

But $$-(ax' + by' + cz') = d$$

Hence $$d = u.\,\overline{P'O} = \pm \mid P'O \mid,$$

according as u and $\overline{P'O}$ have the same or the opposite directions. Thus, *d is the perpendicular distance of the origin from the plane, and this distance is positive or negative according as the vectors u and $\overline{P'O}$ have the same or the opposite directions.*

Take a straight line through a point $P_0 = (x_0, y_0, z_0)$ and a plane given by

$$x = x_0 + \rho\xi, \quad y = y_0 + \rho\eta, \quad z = z_0 + \rho\zeta$$

and $$ax + by + cz + d = 0$$

The point of intersection P_1 of the straight line and the plane will then be given by the value of ρ satisfying the equation

$$(ax_0 + by_0 + cz_0 + d) + \rho(a\,\xi + b\,\eta + c\,\zeta) = 0$$

(1) If $a\,\xi + b\,\eta + c\,\zeta \neq 0$, there is a definite value of ρ which is to be substituted in the equations of the straight line to obtain the point P_1 and therefore the distance $\mid P_0P_1 \mid$. In particular, if $(ax_0 + by_0 + cz_0 + d) = 0$, so that (x_0, y_0, z_0) is a point of the plane, we have $\rho = 0$.

(2) If $a\,\xi + b\,\eta + c\,\zeta = 0$, the value of ρ is undetermined. It is evident that in this case the straight line is parallel to the plane unless $ax_0 + by_0 + cz_0 + d = 0$. If moreover $ax_0 + by_0 + cz_0 + d = 0$, the straight line lies on the plane.

Consider now a plane and a vector normal to the plane. With reference to the direction of this normal vector, we shall associate with the plane a positive sense of rotation in the plane defined thus : *The direction of the normal shall bear to the positive sense of rotation the same relation as the direction of the translational motion of a right-handed screw bears to its positive sense of rotation.* We shall then connect, for the sake of convenience, with each of the two Hessian normal forms of the equation of the plane, an orientation in the plane such that the direction of the normal given by the coefficients and the orientation result in a right-handed screw. Such a plane shall be called an *oriented plane*.

Let $ax+by+cz+d = 0$ be the equation, in Hessian normal form, of an oriented plane, $P_0 = (x_0, y_0, z_0)$ be an arbitrary point of the space and $P_1 = (x_1, y_1, z_1)$ the foot of the perpendicular from P_0 to the plane. The two vectors $\overline{P_1P_0}$ and (a, b, c) are then parallel, and so their scalar product gives the distance of P_0 from the given oriented plane. This distance is an algebraic quantity and is given by

$$a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1) = ax_0+by_0+cz_0+d$$

Thus, *if $ax+by+cz+d = 0$ is the equation in Hessian normal form of an oriented plane, the perpendicular distance of an arbitrary point P_0 from the plane is obtained by substituting the coordinates of P_0 for x, y, z in the expression $ax+by+cz+d$.*

The angle between two oriented planes is defined as the angle between their normals.

Finally, it follows from (12.14) that the equation of a plane passing through three noncollinear points (x_i, y_i, z_i), $i = 1, 2, 3$, can be put as the vanishing of a determinant :

$$| x\, y_1\, z_2\, 1 | = 0 \tag{12.16}$$

The equations (12.12), (12.13), (12.14), (12.15), (12.16) may be compared with (1.5), (1.9), (1.6), (1.7), (1.8) respectively.

48.1 Vector product of two vectors. Let $v_1 = (\xi_1, \eta_1, \zeta_1)$ and $v_2 = (\xi_2, \eta_2, \zeta_2)$ be two nonparallel vectors, and let the vector (ξ, η, ζ) be orthogonal to both the vectors v_1 and v_2. Then

$$\xi_1\xi+\eta_1\eta+\zeta_1\zeta = 0, \qquad \xi_2\xi+\eta_2\eta+\zeta_2\zeta = 0$$

Therefore $$\xi : \eta : \zeta : = \begin{vmatrix} \eta_1 & \zeta_1 \\ \eta_2 & \zeta_2 \end{vmatrix} : \begin{vmatrix} \zeta_1 & \xi_1 \\ \zeta_2 & \xi_2 \end{vmatrix} : \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix}$$

The vector whose coordinates are the expressions on the right-hand side is called the *vector product* of the two vectors v_1 and v_2, and shall be denoted by the notation $v_1 \times v_2$. Thus,

$$v_1 \times v_2 = (\eta_1\zeta_2 - \zeta_1\eta_2,\ \zeta_1\xi_2 - \xi_1\zeta_2,\ \xi_1\eta_2 - \eta_1\xi_2) \tag{12.17}$$

The Hamiltonian notation for the vector product is $V v_1 v_2$. The vector $v_1 \times v_2$ has the following properties :

(1) It is orthogonal to both v_1 and v_2 and is therefore normal to any plane which is parallel to both v_1 and v_2.

(2) The product is noncommutative. It is evident from the definition that the vectors $v_1 \times v_2$ and $v_2 \times v_1$ have the same length but opposite directions. So we may write

$$| v_1 \times v_2 | = | v_2 \times v_1 |, \quad v_1 \times v_2 = -v_2 \times v_1.$$

(3) The direction of $v_1 \times v_2$ is uniquely defined by v_1 and v_2. For example, if v_1 and v_2 are unit vectors in the directions of the positive x- and y- axes respectively, so that $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, then $v_1 \times v_2$ has the coordinates $(0, 0, 1)$. That is, $v_1 \times v_2$ is the unit vector in the direction of the positive z- axis. In general, if v_1 and v_2 are orthogonal vectors, then v_1, v_2 and $v_1 \times v_2$ form a right-handed system through a given point ; hence if we put $v_1 \times v_2 = v_3$, then $v_2 \times v_3 = v_1$, $v_3 \times v_1 = v_2$.

(4) If the unit of length changes, *i.e.*, if we multiply v_1 and v_2 by a constant factor c, then $v_1 \times v_2$ is multiplied by c^2. As a matter of fact, if $v_1 = \overline{P_3P_1}$ and $v_2 = \overline{P_3P_2}$, then the equation (12.6) giving the area of the triangle $P_1P_2P_3$ can be written as

$$2\Delta = | v_1 \times v_2 |$$

And if two skew lines p_1 and p_2 are parallel to the vectors v_1 and v_2 and P_1, P_2 are points of p_1, p_2 respectively, then the equation (12.11) giving the shortest distance between p_1 and p_2 can be written as

$$d = \left| \frac{\overline{P_1P_2}.\ (v_1 \times v_2)}{| v_1 \times v_2 |} \right|$$

49. Intersection of planes. I. Consider the two planes

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

(1) If the rank of the matrix

$$M_1 = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

is two, the planes intersect in a straight line. In this case, if (x_0, y_0, z_0) is a point common to both the planes, we get

$$a_1x_0 + b_1y_0 + c_1z_0 + d_1 = 0$$
$$a_2x_0 + b_2y_0 + c_2z_0 + d_2 = 0$$

Subtracting from the corresponding equations of the planes, we have

$$a_1(x-x_0) + b_1(y-y_0) + c_1(z-z_0) = 0$$
$$a_2(x-x_0) + b_2(y-y_0) + c_2(z-z_0) = 0$$

Solving for the ratios of $x-x_0$, $y-y_0$, $z-z_0$, we obtain

$$\frac{x-x_0}{b_1c_2-c_1b_2} = \frac{y-y_0}{c_1a_2-a_1c_2} = \frac{z-z_0}{a_1b_2-b_1a_2} \qquad (12.18)$$

as the equations of the line of intersection of the two planes.

(2) If the rank of M_1 is one, two cases may arise according as the rank of the following matrix is two or one :

$$M_2 = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$$

(*i*) If the rank of M_2 is two, the two planes are parallel.

(*ii*) If the rank of M_2 is one, the two planes are coincident.

II. Consider now the three planes

$$a_ix + b_iy + c_iz + d_i = 0, \quad i = 1, 2, 3$$

(1) The three planes will meet in a point if the three linear equations have a solution. The necessary and sufficient condition for this is that the rank of the matrix

$$M_3 = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

is three. The point of intersection is then given by

$$\frac{x}{|\,b\ c\ d\,|} = \frac{y}{|\,a\ d\ c\,|} = \frac{z}{|\,d\ a\ b\,|} = \frac{-1}{|\,a\ b\ c\,|}. \qquad (12.19)$$

(2) If the rank of M_3 is less than three, *i.e.*, if $|\,a\ b\ c\,| = 0$, two cases may arise according as the rank of the matrix M_1 is two or one :

(*i*) If the rank of M_1 is two, the three planes are parallel to one straight line and may form a prism provided that the rank of the matrix

$$M_4 = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

is three. If the rank of M_4 is also two, the three planes intersect in a straight line.

(*ii*) If the rank of M_3 is one, the three planes are parallel provided the rank of M_4 is two. If the rank of M_4 is one, the three planes are coincident.

III. Consider the following equations of four planes and the matrix of their coefficient :

$$a_ix+b_iy+c_iz+d_i=0, \quad i=1, 2, 3, 4,$$

$$M_5 = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

Among the different cases that may arise, we notice the following :

(1) If the rank of M_5 is three, the planes meet in a point or have no common point.

(2) If the rank of M_5 is two, the planes have a line in common or have no common point.

(3) If the rank of M_5 is one, the planes are coincident.

50. The tetrahedron. A *tetrahedron* is a figure formed by four noncoplanar points, no three of which are therefore collinear. The four points are called the *vertices*, the six lines joining every pair of the vertices are called the *sides*, and the four planes, each passing through three of the vertices, are called the *faces* of the tetrahedron.

Let $P_i=(x_i, y_i, z_i)$, $i=1, 2, 3$, be three distinct points. If $ax+by+cz+d=0$ is the equation of the plane passing through these three points, we have the three equations

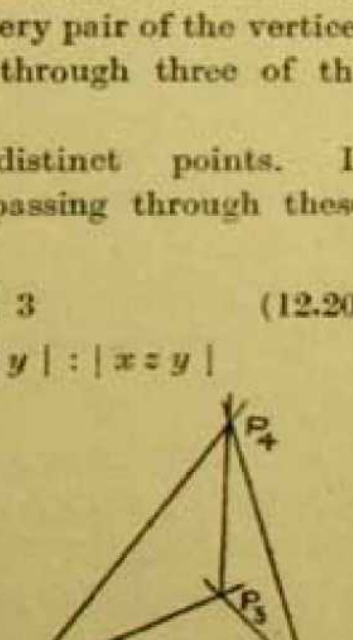
$$ax_i+by_i+cz_i+d=0, \quad i=1, 2, 3 \tag{12.20}$$

whence $\quad a:b:c:d = |\,y\,z\,1\,| : |\,x\,1\,z\,| : |\,1\,x\,y\,| : |\,x\,z\,y\,|$

So, $\quad a\,|\,x\,y\,z\,| + d\,|\,y\,z\,1\,| = 0$

$\quad b\,|\,x\,y\,z\,| + d\,|\,x\,1\,z\,| = 0$

$\quad c\,|\,x\,y\,z\,| + d\,|\,1\,x\,y\,| = 0$



Two cases may arise :

(*i*) If $|\,x\,y\,z\,| \neq 0$, the three determinants $|\,y\,z\,1\,|$, $|\,x\,1\,z\,|$, $|\,1\,x\,y\,|$ cannot all be zero. So the ratios a/d, b/d, c/d are known and the plane is uniquely determined.

(*ii*) If $|\,x\,y\,z\,| = 0$, either the determinants $|\,y\,z\,1\,|$, $|\,x\,1\,z\,|$, $|\,1\,x\,y\,|$ are all zero or only $d=0$. In the first alternative when all the determinants are zero, the plane is undetermined. If however $d = 0$, the three equations (12.20) form a system of linear homogeneous equations in a, b, c and solutions for a, b, c, other than all zero, exist; therefore the plane is uniquely determined. Thus, the plane passing through the three points P_1, P_2, P_3 is uniquely determined unless the determinants $| y\, z\, 1 |$, $| x\, 1\, z |$, $| 1\, x\, y |$ all vanish, that is, unless the area of the triangle $P_1P_2P_3$ vanishes, that is, unless P_1, P_2, P_3 are collinear.

Now, let $ax+by+cz+d = 0$ be the plane through P_1, P_2, P_3, these points being noncollinear. So,

$$a = \rho\, | y\, z\, 1 |, \quad b = \rho\, | x\, 1\, z |, \quad c = \rho\, | 1\, x\, y |, \quad d = -\rho\, | x\, y\, z |$$

Therefore $$| \rho | = \left| \frac{\sqrt{(a^2+b^2+c^2)}}{\sqrt{\{ | y\, z\, 1 |^2 + | x\, 1\, z |^2 + | 1\, x\, y |^2 \}}} \right|$$

Let $P_4 = (x_4, y_4, z_4)$ be a point external to the plane. If δ is the perpendicular distance of P_4 from the plane, Δ the area of the triangle $P_1P_2P_3$ and V the volume of the tetrahedron $P_1P_2P_3P_4$, all given absolute values, then

$$3V = \delta\, \Delta.$$

Now $$\delta = | (ax_4+by_4+cz_4+d) : \sqrt{(a^2+b^2+c^2)} |$$

$$2\Delta = | \sqrt{\{ | y\, z\, 1 |^2 + | x\, 1\, z |^2 + | 1\, x\, y |^2 \}} |$$

$$= | \sqrt{(a^2+b^2+c^2)} : \rho |$$

Therefore

$$6V = | (ax_4+by_4+cz_4+d) : \rho |$$

Or, substituting the values of a, b, c, d,

$$6V = | \, | x\, y\, z\, 1 | \, | \tag{12.21}$$

In particular, if P_4 coincides with the origin O, six times the volume of the tetrahedron $P_1P_2P_3O$ is

$$| d : \rho | = | \, | x\, y\, z | \, |$$

It is seen that this volume vanishes when $d = 0$, *i.e.*, when the plane through P_1, P_2, P_3 passes through the origin. This may also be seen from (12.16). Formula (12.21) may be compared with (1.10).

Some properties of tetrahedron. Let $P_i = (x_i, y_i, z_i)$, $i = 1, 2, 3, 4$, be the vertices of a tetrahedron.

(1) Let C be the centroid of the triangle $P_1P_2P_3$. Then the coordinates of C are

$$((x_1+x_2+x_3)/3, \quad (y_1+y_2+y_3)/3, \quad (z_1+z_2+z_3)/3)$$

The coordinates of the point P on the line P_4C such that

$| P_4P | : | PC | = 3 : 1$ are easily seen to be

$$\left((x_1+x_2+x_3+x_4)/4, \quad (y_1+y_2+y_3+y_4)/4, \quad (z_1+z_2+z_3+z_4)/4 \right)$$

These coordinates show that lines joining the vertices of a tetrahedron with the centroids of the opposite faces all go through a point P, which divides each of them in the ratio 3 : 1. Also, the lines joining the mid-points of the opposite edges all go through the same point P.

(2) The six planes, each passing through an edge and bisecting the opposite edge, cannot evidently have a line in common. For if they had, this line would be coplanar with all the edges. The equations of the plane through the line P_1P_2 and through the mid-point of the segment P_3P_4 can be written, by (12.13), as

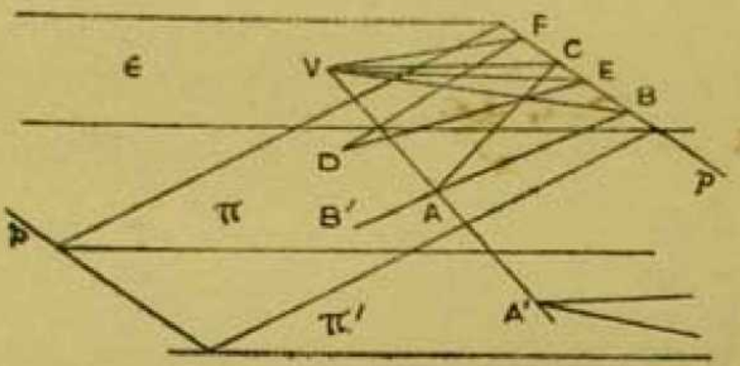
$$\begin{aligned} x &= \lambda x_1 + \mu x_2 + \nu\ (x_3+x_4)/2 \\ y &= \lambda y_1 + \mu y_2 + \nu\ (y_3+y_4)/2 \qquad \lambda+\mu+\nu = 1 \\ z &= \lambda z_1 + \mu z_2 + \nu\ (z_3+z_4)/2 \end{aligned}$$

Similarly for the equations of the other five planes. If these six planes have a point in common, it must be possible to choose for each of these six sets of equations values of the constants such that they would give the same set of values for x, y, z. Putting $\lambda = \mu = 1/4, \nu = 1/2$ in the above and similarly in the other sets of equations, we see that the six planes meet in the same point as obtained in (1) above.

51. Projection. Take two planes π and π' and a point V in space. If a straight line through V meets the planes in P and P' respectively, then P (or P') is called the *projection* of P' (or P) on the plane π (or π') ; V is called the *centre of projection*. It is clear that the projection of a plane figure onto another plane is, in general, a figure. In particular, collinear points are projected into collinear points and concurrent straight lines into concurrent straight lines. In the special case when the straight line VP joining a point P on π is parallel to π', the point P' does not exist ; the point P is then said to be a *vanishing point* of π. All the vanishing points of π then lie on a straight line p, namely, the line of intersection of π and the plane through V parallel to π'; the straight line p whose projection onto π' does not exist is said to be the *vanishing line* of π. Similarly there may be vanishing points and vanishing line on π'. It is evident that if the planes π, π' intersect in a straight line s, the vanishing lines of π and π' are both parallel to s. The straight line s is called the *axis of projection* and is such that a straight line and its projection intersect the axis in the same point.

Let the two arms AB and AC of an angle $\sphericalangle BAC$ in the plane π meet the vanishing line of π in the points B and C. The $\sphericalangle BAC$ is projected into an angle whose magnitude is equal to that of the angle BVC. For, the projections of the lines AB and AC are respectively parallel to the lines VB and VC.

To project a plane π containing a given line p in such a manner that p becomes the vanishing line and at the same time two given angles in π are projected into angles of given magnitudes onto a plane π' properly chosen.



Construction : Through p draw any plane ϵ and let the plane of projection π' be taken parallel to ϵ. Consider first the case when the two pairs of arms AB, AC and DE, DF of the given angles $\sphericalangle BAC$ and $\sphericalangle EDF$ in π meet p in the pairs of points B, C and E, F. Suppose that these two angles are to be projected into angles of magnitudes φ and θ respectively, each less than 180°. On the segments BC and EF in the plane ϵ describe on the same side of p segments of circles containing angles φ and θ respectively and let the two segments of circles intersect in a point V. Now if V is taken as the centre of projection, then the line p is made the vanishing line and at the same time $\sphericalangle BAC$ and $\sphericalangle EDF$ are projected into angles of magnitudes φ and θ respectively. Secondly, if one of the arms AB does not meet p, then either the half-ray AB' opposite to AB meets p or the line AB is parallel to p. If the half-ray AB' meets p in B', we describe on $B'C$ segment of a circle containing the angle $180° - \varphi$. If the line AB is parallel to p, we draw in the plane ϵ the line CV such that the angle between CV and the half-ray of p in the same direction as the arm AB is $180° - \varphi$. The centre of projection V is then determined as the intersection of CV and the segment of circle on EF.

The construction fails if the segments of the circles do not intersect in a point outside p.

Cor. 1. *Any triangle can be projected into an equilateral triangle.* For, if we project two of its angles into angles of 60°, the third angle will also be projected into an angle of 60°. In the construction p can be so chosen that the circles intersect.

Cor. 2. *Any plane quadrilateral may be projected into a square.* Let $ABCD$ be the quadrilateral whose diagonals AC, BD meet in E. Also, let AB, CD meet in F and AD, BC meet in G. Now project the quadrilateral such that the line FG is made the vanishing line and at the same time $\sphericalangle\ BAG$ and $\sphericalangle\ BEA$ are projected into right angles. This is possible, for the arms of these angles intersect the line FG in two pairs of points separating one another. The quadrilateral will then be projected into a square. For, by making FG the vanishing line, the projected figure is made a parallelogram; the projection of $\sphericalangle\ BAG$ into a right angle makes this parallelogram a rectangle; and finally the projection of $\sphericalangle\ BEA$ into a right angle ensures that the rectangle is a square.

CHAPTER XIII

PROJECTIVE SPACE

52. Principal notions. From chapter IX onwards in plane geometry the Euclidean plane had been extended to a projective plane and it was shown that a projective plane could be represented by homogeneous coordinates. We shall now introduce homogeneous coordinates for the space.

Let us define a *projective point* by four numbers (x_1, x_2, x_3, x_4), called its coordinates, not vanishing simultaneously and let two projective points be considered identical if and only if their coordinates are proportional. Hence a common factor $\rho \neq 0$ of the coordinates remains indefinite and so a projective point is defined by

$$\rho\,(x_1, x_2, x_3, x_4), \quad \rho \neq 0, \quad (x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0) \tag{13.1}$$

As in plane geometry, the coordinates $(x_1, x_2, x_3\ x_4)$ shall be briefly denoted by (x_i). We shall require the following matrices :

$$N_1 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} a_1' & a_2' & a_3' & a_4' \\ b_1' & b_2' & b_3' & b_4' \end{pmatrix},$$

$$N_3 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1' & a_2' & a_3' & a_4' \\ b_1' & b_2' & b_3' & b_4' \end{pmatrix}, \qquad N_4 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix},$$

$$N_5 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}, \qquad N_6 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} \tag{13.1'}$$

If (a_i) and (b_i) are two projective points, the rank of the matrix N_1 is different from zero. If the rank of N_1 is one, the points are identical and if it is two, the points are distinct. Let the points (a_i), (b_i) be distinct ; then the points

$$(\gamma a_i + \lambda b_i), \quad (\gamma, \lambda) \neq (0, 0) \tag{13.2}$$

are said to form a *row* (or *range*) *of points* determined by (a_i) and (b_i) ; points of a row are said to be collinear. Let (a_i') and (b_i') be two distinct points of the row (13.2) ; then the rank of each of the matrices N_2 and N_3 is two. Therefore (a_i) and (b_i) belong to the row of points $(\gamma a_i' + \lambda b_i')$ determined by (a_i') and (b_i'). Hence the two rows are identical. Thus, *every row is uniquely determined by every pair of distinct points belonging to it.*

Let (c_i) be a point not belonging to (13.2) ; then the rank of the matrix N_4 is three. In this case, the points

$$(\gamma a_i + \lambda b_i + \mu c_i), \qquad (\gamma, \lambda, \mu) \neq (0, 0, 0) \tag{13.3}$$

are said to form a *plane field* ; points of a plane field are said to be coplanar. A matrix formed by the coordinates of any number of coplanar points has a rank which cannot be greater than three ; and if there are three noncollinear points among them, the rank is three. It therefore follows, as above, that *every plane field is uniquely determined by every triplet of noncollinear points belonging to it.*

Let (x_i) be a point of the field (13.3) ; then the rank of each of the matrices N_4 and N_5 is three. Therefore

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix} = 0,$$

where the cofactors u_i of x_i, $i = 1, 2, 3, 4$, in the above determinant cannot all be zero. Hence

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0, \quad (u_1, u_2, u_3, u_4) \neq (0, 0, 0, 0) \tag{13.4}$$

On the other hand, every solution (x_1, x_2, x_3, x_4) of (13.4) is linearly dependent on (a_i), (b_i), (c_i) ; this means that the solution represents a point of the plane field (13.3). Moreover, since N_4 is of rank three, the system of homogeneous linear equations

$$\sum a_i x_i = 0, \quad \sum b_i x_i = 0, \quad \sum c_i x_i = 0$$

has only one homogeneous solution, *i.e.*, a solution defined up to a non-zero common factor. Hence *to every plane field there corresponds an equation (13.4) in which the numbers u_i are uniquely given except for an arbitrary common factor $\sigma \neq 0$.*

A system of four numbers given by

$$\sigma(u_1, u_2, u_3, u_4), \quad \sigma \neq 0, \quad (u_1, u_2, u_3, u_4) \neq (0, 0, 0, 0) \qquad (13.5)$$

is said to define a *projective plane*. Thus we have now two kinds of mathematical entities (13.1) and (13.5), namely the points and the planes, both represented by four (homogeneous) coordinates in the same manner. By (13.4) a correspondence is established between plane fields and projective planes in such a manner that the points of a plane field lie on the corresponding projective plane.

It will be advantageous to use the word 'incident' in the usual sense of relationship between the three kinds of entities called points, lines and planes. Any two entities of different kinds are said to be *incident* with each other when one lies on or passes through the other. Thus, collinear (coplanar) points are points which are incident with the same line (plane).

The equation (13.4) can then be considered as the *condition of incidence* of the points (x_i) and the plane (u_i). Such an equation as (13.4) may sometimes be considered as a linear homogeneous equation in x_i with coefficients u_i and sometimes as a linear homogeneous equation in u_i with coefficients x_i. To solve a system of equations of this type, let us consider the system

$$a_{k_1}x_1 + a_{k_2}x_2 + a_{k_3}x_3 + a_{k_4}x_4 = 0, \quad k = 1, 2 \ldots ,$$

where the coefficients a_{ki} form a matrix of rank $m \leq 4$. It is known from the basic algebra that every solution (x_1, x_2, x_3, x_4) of these equations is also a solution of $\sum u_i x_i = 0$, where (u_1, u_2, u_3, u_4) is any '4-vector' of the 'vector space' U of rank m generated by the '4-vectors' $(a_{k_1}, a_{k_2}, a_{k_3}, a_{k_4})$. The solutions, on the other hand, form a 'vector space' X of rank $4-m$; and the connection between U and X is reciprocal. The terms and the notions are, of course, used in the sense of algebra.

Let us apply the above algebraical facts in our geometry and take into consideration the cases $m = 4, 3, 2, 1, 0$. All these cases are given in the following scheme which can be read from the left to the right or from the right to the left.

Rank m	Name of the set of planes forming a vector space U	Name of the set of points forming a vector space X	Rank $4-m$
4	The field of all planes	——————	0
3	A bundle of planes	A point	1
2	A pencil of planes	A row of points	2
1	A plane	A plane field	3
0	——————	The field of all points	4

Read from the left to the right, the scheme means that given a set of planes whose coordinates form a vector space U of rank m, there exists a set of points whose coordinates form a vector space X of rank $4-m$ such that every point of X is incident with every plane of U. Reading from the right to the left, we obtain a system U of rank m as 'solutions' of a system X of rank $4-m$ and thus get those set of planes which are incident with a given set of points. It is to be noticed that X is composed of points and U of planes. The names of the vector spaces U and X are given respectively in the second and third columns of the scheme. They are *elementary geometric forms* of different dimensions (see §35) and are defined as follows :

The field of all planes consists of all planes of the projective space and *the field of all points* consists of all points of the projective space ; they are *three-dimensional geometric forms. A bundle* (or *sheaf*) *of planes* is the set of all planes that are incident with a point, and *a plane field* is the set of all points that are incident with a plane ; they are *two-dimensional geometric forms. A pencil of planes* is the set of all planes that are incident with a line and *a row of points* is the set of all points that are incident with a line ; they are *one-dimensional geometric forms.*

It follows immediately from the above description that, as in the projective plane, there is *duality* in the projective space. This duality is obtained by interchanging 'point' and 'plane' *i.e.*, (x)-coordinates and (u)-coordinates. The condition of incidence $\sum u_i x_i = 0$ remains unaltered by this duality. Given the above scheme, we obtain the dual scheme by first interchanging the two middle columns and then interchanging the second and the sixth rows, the third and the fifth rows of each of these two columns. Thus a bundle of planes and a plane field are dual forms. For every theorem based on the concepts of projective point, projective-plane and incidence there exists a dual theorem, and the interchange of the notations x and u cannot convert a true theorem into a false one. For every property that may be derived from the three fundamental concepts there exists a dual property and a dictionary may be compiled for translation of every projective property or theorem into its dual. Rows of points and pencils of planes are, for example, dual notions; but they are also connected by the fact that to each particular row of points there corresponds a particular pencil of those planes which are incident with every point of the row, and conversely. The row and the pencil connected in this way give rise to a new mathematical entity called the *projective line*. Therefore, a line is *dual to itself*. Every individual line can be determined by a particular row, say $(\gamma x_i + \lambda y_i)$, as well as by a particular pencil of planes, say $(\gamma u_i + \lambda v_i)$. When it is determined by a row, the line is called the *base* of the row; and when determined by a pencil, it is called the *axis* of the pencil.

Two distinct rows cannot have more than one common point, because a row is uniquely defined by two points. If they have a common point, this point is called the point of intersection of the two lines which are the bases of the two rows. When two rows have a common point, they cannot have more than three independent points; the rows then belong to the same plane field, *i.e.*, there exists a plane with which both the rows are incident. By consideration of duality it is seen that if two pencils of planes are distinct, they may have at most one common plane which is then called the plane passing through the two lines which are the axes of the pencils. In this case the pencils belong to the same bundle. Two lines therefore intersect if and only if they are incident with the same plane.

Consider the points in which a line $(\gamma x_i + \lambda y_i)$ intersects a plane (u_i). For these points we must have

$$\gamma l_1 + \lambda l_2 = 0, \quad \text{where} \quad l_1 = \sum u_i x_i, \quad l_2 = \sum u_i y_i$$

This equation is a linear and homogeneous equation in γ, λ, the coefficients l_1 and l_2 being given numbers. If the rank of the one-rowed matrix $(l_1\ l_2)$ is one, there is one solution, *i.e.*, there is one point of intersection; and if the rank is zero, every γ, λ satisfy the equation, *i.e.*, the line lies in the plane. We may now state the following dual results :

Two points define one and only one line.	Two planes define one and only one line.
Three noncollinear points define one and only one plane.	Three noncoaxal planes define one and only one point.
If a line g does not lie in a plane α, g and α define one and only one point incident with both.	If a line g does not pass through a point A, g and A define one and only one plane incident with both.
Two lines incident with the same point are incident with one and only one plane.	Two lines incident with the same plane are incident with one and only one point.

52.1. Projective plane and plane projective geometry. The formulae of the preceding article remind us of those of the plane projective geometry. Indeed, by dropping the last coordinate of the projective points of the projective space we get the projective points of the plane geometry and similarly we get the projective lines of the plane geometry out of the coordinates of the projective planes of the space.

Specially there exists a one-to-one correspondence between the points (x_1, x_2, x_3) of the plane projective geometry and the points $(x_1, x_2, x_3, 0)$ of the plane field V, say, which, by (13.4), may be identified with the plane $V = (0, 0, 0, 1)$. Of course, nowhere can the coordinates vanish simultaneously. Now the lines in V are the axes of pencils of planes where each pencil has V as a plane ; therefore the planes of these pencils have the coordinates

$$(\gamma u_1, \gamma u_2, \gamma u_3, \gamma u_4 + \lambda) \quad \text{where} \quad (u_1, u_2, u_3) \neq (0, 0, 0)$$

Hence we may denote such a pencil by

$$\gamma\,(u_1, u_2, u_3, *\,), \quad \text{where } * \text{ takes all values.}$$

A point of the axis of this pencil must satisfy the conditions

$$x_1 u_1 + x_2 u_2 + x_3 u_3 = 0, \quad x_4 = 0$$

If therefore we set up the correspondence

$$\rho\,(x_1, x_2, x_3, 0) \longrightarrow \rho\,(x_1, x_2, x_3), \quad \sigma\,(u_1, u_2, u_3, *) \longrightarrow \sigma\,(u_1, u_2, u_3),$$

the points and the lines of V are represented by the points and the lines of the plane projective geometry in such a manner that a pair of incident point and line of V is represented by a pair of incident point and line of the plane projective geometry. That is to say, 'the plane projective geometry' holds in the plane V of the projective space.

53. Projective space as an extension of the Euclidean space. Let Σ denote a projective space and $V = (0, 0, 0, 1)$ be the plane of Σ as defined in the last article. Those points of Σ which are not situated on V can be represented by $(x_1, x_2, x_3, 1)$, where the first three coordinates x_1, x_2, x_3 are uniquely determined for each of these points without any common arbitrary factor. We may therefore set up a one-to-one correspondence between the quadruplets $(x_1, x_2, x_3, 1)$ and the triplets (x_1, x_2, x_3). Since the triplets are uniquely determined without any common factor, they may be taken to represent points of a Euclidean space S. Again, for the planes (u_1, u_2, u_3, u_4) of Σ other than V, we must have $(u_1, u_2, u_3) \neq (0, 0, 0)$. Therefore there exists simultaneously a one-to-one correspondence between the planes, other than V, of Σ and the planes of S in such a manner that corresponding planes have the same (four) coordinates. It follows from the nature of these correspondences that if $u_1x_1 + u_2x_2 + u_3x_3 + u_4 = 0$, *i.e.*, if a point and a plane, other than V, of Σ are incident with one another, the corresponding point and the corresponding plane of S are also incident, and conversely. We may therefore *identify* S with that portion of Σ which is outside V and regard Σ as an *extension* of S; this extension is obtained by adding, so to say, the plane V to the space S. We now say that V is *the plane at infinity* (or *the ideal plane*) for S and the points and lines of V are *the points and lines at infinity* for S. Of course, these elements at infinity do not belong to S.

Consider now two points $(x_1, x_2, x_3, 1)$ and $(y_1, y_2, y_3, 1)$ of Σ. The row generated by the two points is, by (13.2),

$$(\gamma x_1+\lambda y_1,\quad \gamma x_2+\lambda y_2,\quad \gamma x_3+\lambda y_3,\quad \gamma+\lambda),\quad (\gamma, \lambda) \neq (0, 0) \qquad (13.6)$$

As a common factor of γ, λ is arbitrary, this row consists of the following points of S

$$x = \gamma x_1+\lambda y_1,\quad y = \gamma x_2+\lambda y_2,\quad z = \gamma x_3+\lambda y_3,\quad \gamma+\lambda = 1$$

and a point at infinity (taking $\gamma+\lambda = 0$)

$$\lambda\,(a_1, a_2, a_3, 0),\quad \text{where}\quad a_i = y_i - x_i,\ i = 1, 2, 3 \qquad (13.6')$$

The points of S which belong to the row (13 6) can now be represented (since $\gamma+\lambda = 1$) by

$$x = x_1+\lambda a_1,\quad y = x_2+\lambda a_2,\quad z = x_3+\lambda a_3 \qquad (13.6'')$$

It therefore follows from (12.7) that the points (13.6″) define a straight line which passes through the point (x_1, x_2, x_3) and is parallel to the vector $v = (a_1, a_2, a_3)$. But except for a numerical factor λ, this vector

v is uniquely given by the point at infinity (13.6′) of the row (13.6). This shows that *parallel straight lines in S are those which have a common point in the plane at infinity V when S is extended to Σ.*

Consider further a third point $(z_1, z_2, z_3, 1)$ of Σ which is not collinear with the two given points. Then the plane field W generated by the three points is, by (13.3),

$$(\gamma x_1+\lambda y_1+\mu z_1,\quad \gamma x_2+\lambda y_2+\mu z_2,\quad \gamma x_3+\lambda y_3+\mu z_3,\quad \gamma+\lambda+\mu), \qquad (13.7)$$
$$(\gamma, \lambda, \mu) \neq (0, 0, 0)$$

As a common factor of γ, λ, μ is arbitrary, we have here also to consider, as above, the two cases : $\gamma+\lambda+\mu = 0$ and $\gamma+\lambda+\mu = 1$. In the first case we get the row

$$(\lambda a_1+\mu b_1,\quad \lambda a_2+\mu b_2,\quad \lambda a_3+\mu b_3, 0), \text{ where further } b_i = z_i - x_i \qquad (13.7')$$

and in the second case we get the plane

$$x = x_1+\lambda a_1+\mu b_1,\quad y = x_2+\lambda a_2+\mu b_2,\quad z = x_3+\lambda a_3+\mu b_3 \qquad (13.7'')$$

The points (13.7′) furnish a row in V, *i.e.*, they give the points at infinity on a line at infinity ; the base of this row is the line of intersection of the planes W and V. To every point of this row there corresponds a system of vectors $w = (\lambda a_1+\mu b_1,\ \lambda a_2+\mu b_2,\ \lambda a_3+\mu b_3)$, depending on the numerical factors λ, μ. The plane (13.7″) is a plane of S ; and it follows from (12.12) that every point of this plane is obtained by attaching the vectors w to the point (x_1, x_2, x_3). Therefore *parallel planes in S are those which have a common line in V when S is extended to Σ.*

54. Pluecker coordinates of a line. Before leaving the present discussion we may give a brief description of a special system of line coordinates. Let (a_i) and (b_i) be the homogeneous coordinates of two distinct points of a line g. From the matrix N_1 defined in (13.1′) form the six determinants

$$p_{12} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},\quad p_{13} = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix},\quad p_{14} = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix},$$
$$p_{23} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix},\quad p_{42} = \begin{vmatrix} a_4 & a_2 \\ b_4 & b_2 \end{vmatrix},\quad p_{34} = \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} \qquad (13.8)$$

Obviously, $\qquad p_{ij} = -p_{ji}, \quad i \neq j = 1, 2, 3, 4$

The six quantites p_{ij} thus defined, which cannot be all zero, are called *Pluecker coordinates* of the line g. Since the matrix N_4, given in (13.1′), has its determinant identically equal to zero, it follows that these coordinates satisfy the identical relation

$$p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23} = 0 \qquad (13.9)$$

Pluecker coordinates are homogeneous. For, if

$$\gamma a_i + \lambda b_i \quad \text{and} \quad \mu a_i + \nu b_i$$

are any two points of g and we form the six determinants

$$q_{ij} = \begin{vmatrix} \gamma a_i + \lambda b_i & \gamma a_j + \lambda b_j \\ \mu a_i + \nu b_i & \mu a_j + \nu b_j \end{vmatrix}, \qquad \text{then} \quad q_{ij} = (\gamma\mu - \lambda\nu)\, p_{ij}$$

Let (a_i') and (b_i') be the coordinates of two distinct points of another line g' and let p'_{ij} be the Pluecker coordinates of g'. Then, considering the matrix N_3 as defined in (13.1′), it follows that the lines g and g' are coplanar if and only if det $N_3 = 0$. Therefore, *the necessary and sufficient condition that two lines whose Pluecker coordinates are p_{ij} and p'_{ij} be coplanar is that*

$$\sum p_{ij}\, p'_{kl} \equiv p_{12}p'_{34} + p_{13}p'_{42} + p_{14}p'_{23} + p_{23}p'_{14} + p_{42}p'_{13} + p_{34}p'_{12} = 0 \qquad (13.10)$$

It is understood, in the summation notation on the left, that when the pair of indices ij take the values 12, 13, 14, 23, 42, 34, the pair of indices kl take respectively the complementary values 34, 42, 23, 14, 13, 12.

A line l with coordinates $\gamma p_{ij} + \gamma' p'_{ij}$ is said to be linearly dependent on the lines g and g'. Since (13.9) must also be satisfied by the coordinates of l, we must have

$$\gamma\gamma' \sum p_{ij}\, p'_{kl} = 0$$

If the lines g and g' are skew, *i.e.*, if $\sum p_{ij}\, p'_{kl} \neq 0$, then l is either g or g'. And if g and g' are coplanar, and therefore intersect in a point, say (c_i), different from (a_i), (a'_i), then since

$$\gamma\, p_{ij} + \gamma' p'_{ij} = \lambda \begin{vmatrix} c_i & c_j \\ a_i & a_j \end{vmatrix} + \lambda' \begin{vmatrix} c_i & c_j \\ a'_i & a'_j \end{vmatrix} = \begin{vmatrix} c_i & c_j \\ \lambda a_i + \lambda' a'_i & \lambda a_j + \lambda' a'_j \end{vmatrix},$$

the points (c_i) and $(\lambda a_i + \lambda' a'_i)$ are points of l. Therefore *the line l is a line of the pencil of lines determined by the lines g and g'.*

Let us now suppose that g is a variable line so that its Pluecker coordinates $p_{ij} = a_i b_j - a_j b_i$ are also variables. Consider an equation which is linear in p_{ij}

$$\sum a_{ij}\, p_{ij} = 0, \qquad (13.11)$$

where in the summation the pair of indices ij take the values 12, 13, 14, 23, 42, 34. Then the locus of the line g which satisfies (13.11) is called a *linear complex of lines*. Two cases arise :

(1) First suppose that the coefficients a_{ij} are the Pluecker coordinates of a line. If we put

$$a_{ij} = \rho b_{kl}, \quad \rho \neq 0, \qquad (13.12)$$

where the pairs of indices ij and kl take complementary values, then b_{ij} are also the Pluecker coordinates of a line l, say. It follows from (13.10) that *the linear complex (13.11) consists of all lines which intersect the line l.*

(2) Secondly suppose that a_{ij} are not Pluecker coordinates. Put

$$c_{ij} = a_{ij} - a_{ji}, \quad \text{so that} \quad c_{ij} = -c_{ji} \quad \text{and} \quad c_{ii} = 0$$

Then the equation (13.11) of the linear complex can be written in the normal form as

$$\sum_{i,j=1}^{4} c_{ij}\, p_{ij} = 0, \quad |\, c_{ij} \,| \neq 0 \tag{13.13}$$

If now we suppose that (a_i) is a fixed point and $(b_i) = (x_i)$ is a variable point, the equation (13.13) reduces to

$$\sum c_{ij}\, a_i\, x_j = 0$$

This is a linear equation in x_i with coefficients not all vanishing and hence it represents a plane ; further the line g now goes through a fixed point (a_i). Therefore, *the linear complex is a pencil of lines.*

A line is given either as the join of two points or as the intersection of two planes. From this point of view we have also the Pluecker coordinates of intersection defined as follows :

Let (u_i) and (v_i) be two distinct planes intersecting in a line g. Form the six quantities

$$w_{ij} = u_i v_j - u_j v_i, \qquad i \neq j = 1, 2, 3, 4$$

The quantities w_{ij} are the required coordinates of the line g. They are obviously connected by the relation

$$w_{12}w_{34} + w_{13}w_{42} + w_{14}w_{23} = 0.$$

To establish the relation between the coordinates p_{ij} and w_{ij} of the same line g, let (a_i) and (b_i) be two distinct points of g. We then have the four equations

$$\sum u_i\, a_i = 0, \quad \sum u_i\, b_i = 0, \qquad \sum v_i\, a_i = 0, \quad \sum v_i\, b_i = 0$$

From the first and the second pairs of these equations we get

$$u_2 p_{12} + u_3 p_{13} + u_4 p_{14} = 0 \quad \text{and} \quad v_2 p_{12} + v_3 p_{13} + v_4 p_{14} = 0$$

Similarly for the other pairs. Hence we get

$$p_{12} : p_{13} : p_{14} : p_{23} : p_{42} : p_{34} = w_{34} : w_{42} : w_{23} : w_{14} : w_{13} : w_{12} \tag{13.14}$$

Thus the quantities a_{ij} and b_{ij} in (13.12) are dual Pluecker coordinates of the line l and so the equations

$$\sum a_{ij}\, p_{ij} = 0 \quad \text{and} \quad \sum b_{ij}\, w_{ij} = 0$$

represent the same linear complex.

55. Projectivity. Cross-ratio. We have seen in § 29 that a pencil of lines may be obtained by projecting a row from a point not incident with the base of the row and a row of points obtained by taking a section of a pencil by a line not incident with the centre of the pencil. By 'projection of a point from a line' or 'projection of a line from a point' we shall mean the plane which is incident with both the points and the line ; similarly by 'section of a plane by a line' or 'section of a line by a plane' we shall mean the point which is incident with both the plane and the line.

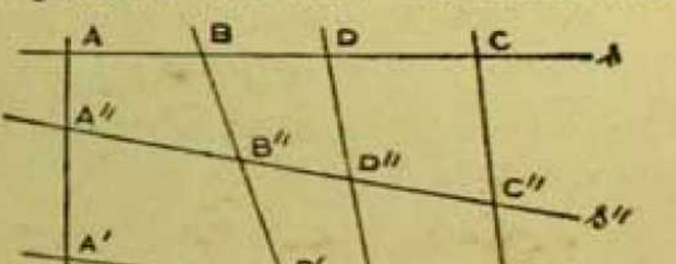
Two rows of points $(ABC\ldots)$, $(A'B'C'\ldots)$ are perspective when they are the sections of one and the same pencil of lines, *i.e.*, when the corresponding points (A, A'), ... of every pair are incident with a line of a pencil of lines. A row $(ABC\ldots)$ and a pencil of lines $(abc\ldots)$ are perspective when the row is a section of the pencil or the pencil is a projection of the row, *i.e.*, when corresponding elements (A, a), ... are those that are incident. A pencil of planes $(\alpha\,\beta\,\gamma\ldots)$ and a row $(ABC\ldots)$ are perspective when the pencil is a projection of the row or the row is a section of the pencil, *i.e.*, when corresponding elements (α, A), ... are those that are incident. A pencil of planes $(\alpha\,\beta\,\gamma\ldots)$ and a pencil of lines $(abc\ldots)$ are perspective when the first is a projection of the second or the second is a section of the first *i.e.*, when corresponding elements (α, a), ... are those that are incident. Two pencils of planes $(\alpha\,\beta\,\gamma\ldots)$, $(\alpha'\,\beta'\,\gamma'\ldots)$ are perspective when they are perspective to the same pencil of lines ; corresponding elements (α, α'), ... are those that are incident with the same line of the pencil of lines. Two pencils of lines $(abc\ldots)$, $(a'\,b'\,c'\ldots)$ are perspective when they are perspective either to the same row or to the same pencil of planes ; corresponding elements (a, a'), ... are those that are incident with the same point of the row in the first case or to the same plane of the pencil in the second case.

The set of all points (and therefore of all lines) that lie on a plane has been said to form a plane field, the plane being the base of the field. A plane field contains an unlimited number of rows of points and the pencils of lines. The set of all planes (and therefore of all lines) that pass through a point has been said to form a bundle, the point being the centre of the bundle. A bundle contains an unlimited number of pencils of lines and pencils of planes. A plane field (α) and a bundle (A) are perspective when the bundle is a projection of the plane field from a point A not incident with the base α or the plane field is a section of the bundle by a plane α not passing through A; corresponding elements are those that are incident. Two plane fields are perspective when they are both perspective to the same bundle; and the two bundles are perspective when they are both perspective to the same plane field.

The row of points, the pencil of lines and the pencil of planes have been called one-dimensional elementary geometric forms whereas the plane field and the bundle have been called two-dimensional elementary geometric forms (§ 52). Two elementary geometric forms of the same dimension are said to be *projective* when they can be connected by a chain of projections and sections, and therefore by a chain of perspectivities. In particular, two perspective forms are projective.

Let us discuss projectivity between two one-dimensional geometric forms. We can always find a geometrical construction establishing the projectivity between two such forms when three pairs of corresponding elements are given. Suppose, for example, it is required to establish the projectivity between two rows ($ABC\ldots.$) and ($A'B'C'\ldots$) when (A, A'), (B, B'), (C, C') are three pairs of corresponding points. Two cases arise according as the bases s and s' of the rows are coplanar or not. The plane construction has been given in § 29. Let then s and s' be skew.

Join AA', BB', CC' and take a point A'' on AA'. The intersection of the plane incident with A'' and BB' and the plane incident with A'' and CC' is a line s'' meeting BB' and CC' in the points B'' and C''. Therefore the pencil of planes s'' ($ABC\ldots.$) and s'' ($A'B'C'\ldots.$) are identical. Hence the pencil s'' ($ABC\ldots.$) is perspective to both the rows ($ABC\ldots.$) and ($A'B'C'\ldots.$). Therefore the rows ($ABC\ldots.$) and ($A'B'C'\ldots.$) are projective. The point D' of ($A'B'C'\ldots.$) which corresponds to a point D of ($ABC\ldots.$) in this projectivity is the point where the plane incident with s'' and D intersects s'.



Again to find a geometrical construction for the projectivity connecting a row and a pencil of planes when three pairs of corresponding elements are given, we take a section of the pencil of planes to obtain a second row. Then by the above construction we find the projectivity between the two rows, as three pairs of corresponding elements are now known. And as the second row is perspective to the given pencil, the required projectivity is established.

We now introduce cross-ratio in one-dimensional geometric forms and show that the cross-ratio remains invariant for projectivity.

Let $$\rho_1(x_i),\ \rho_2(y_i),\ (\gamma x_i+\lambda y_i) = \rho(\xi_i)$$

be any three distinct points of a row. Putting $\gamma x_i = x_i'$, $\lambda y_i = y_i'$, we get the same points represented by $\rho_1'(x_i')$, $\rho_2'(y_i')$, $\rho'(x_i'+y_i')$. Thus any pair of numbers γ, λ can be replaced by 1, 1, and conversely. But we cannot replace simultaneously two such pairs by arbitrary pairs of numbers, as the following consideration will show. Let

$$\begin{aligned}
P_1 &= (x_i) = (x_i'), && \text{where } x_i' = ax_i\\
P_2 &= (y_i) = (y_i'), && \text{where } y_i' = by_i\\
P_3 &= (\xi_i) = (\gamma x_i+\lambda y_i) = (\xi_i'), && \text{where } \xi_i' = c\xi_i\\
P_4 &= (\eta_i) = (\mu x_i+\nu y_i) = (\eta_i'), && \text{where } \eta_i' = d\eta_i
\end{aligned}$$

Then

$$\xi_i = \gamma' x_i'+\lambda' y_i', \qquad \text{where } \gamma' = \frac{c}{a}\gamma,\ \lambda' = \frac{c}{b}\lambda$$

$$\eta_i' = \mu' x_i'+\nu' y_i', \qquad \text{where } \mu' = \frac{d}{a}\mu,\ \nu' = \frac{d}{b}\nu$$

But as $\gamma/\mu \neq \lambda/\nu$, the contention holds.

We *define* cross-ratio of the four points P_1, P_2, P_3, P_4 by

$$(P_1P_2,\ P_3P_4) = \lambda\mu/\gamma\nu = \lambda'\mu'/\gamma'\nu' \tag{13.8}$$

This value is independent of the arbitrary factors a, b, c, d. The same thing holds for 4 planes of a pencil. Let

$$\alpha_1 = (u_i), \qquad \alpha_2 = (v_i), \qquad \alpha_3 = (\gamma u_i+\lambda v_i), \qquad \alpha_4 = (\mu u_i+\nu v_i)$$

be four planes. Then the cross-ratio of the four planes is given by

$$(\alpha_1\alpha_2,\ \alpha_3\alpha_4) = \lambda\mu/\gamma\nu$$

and is independent of the arbitrary factors of the coordinates.

Suppose that the points P_1, P_2, P_3 of the row are points of the Euclidean space S (see §53) ; so we can choose $x_4 = y_4 = 1$ and $\gamma+\lambda = 1$, and then

$$(x_1, x_2, x_3), \qquad (y_1, y_2, y_3), \qquad (\xi_1, \xi_2, \xi_3)$$

can be considered as the (orthogonal) Cartesian coordinates of P_1, P_2, P, respectively. We can write

$$\xi_i = x_i + \lambda (y_i - x_i) ;$$

hence
$$\lambda = \frac{\xi_i - x_i}{y_i - x_i}, \quad \gamma = 1 - \lambda = \frac{y_i - \xi_i}{y_i - x_i}$$

So
$$\frac{\lambda}{\gamma} = \frac{\xi_i - x_i}{y_i - \xi_i} = \frac{\overline{P_1P_3}}{\overline{P_3P_2}}$$

As in § 53, we distinguish two cases :

(*i*) Let P_4 be also a point of S; then we choose $\mu + \nu = 1$, and consider (η_1, η_2, η_3) as the Cartesian coordinates of P_4. We have therefore, as above,

$$\frac{\nu}{\mu} = \frac{\eta_i - x_i}{y_i - \eta_i} = \frac{\overline{P_1P_4}}{\overline{P_4P_2}}$$

Hence
$$\frac{\lambda\mu}{\gamma\nu} = \frac{\overline{P_1P_3}}{\overline{P_3P_2}} \, \frac{\overline{P_4P_2}}{\overline{P_1P_4}} = \frac{\overline{P_1P_3}}{\overline{P_2P_3}} \Big/ \frac{\overline{P_1P_4}}{\overline{P_2P_4}}$$

This result agrees with (2.1).

(*ii*) Let P_4 be a point at infinity ; then $\mu + \nu = 0$, or $\nu/\mu = -1$.

Hence
$$\lambda\mu/\gamma\nu = \overline{P_1P_3} \Big/ \overline{P_2P_3}$$

This result agrees with what has been said in §29.1

Consider again four planes of a pencil and let none of them be the plane at infinity. We can therefore suppose that the equations of the four planes $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are given in Hessian normal forms as

$$l_1(x_1, x_2, x_3) = 0, \qquad l_2(x_1, x_2, x_3) = 0,$$
$$l_3(x_1, x_2, x_3) = \gamma l_1 + \lambda l_2 = 0,$$
$$l_4(x_1, x_2, x_3) = \mu l_1 + \nu l_2 = 0$$

respectively. Let $P = (\xi_1, \xi_2, \xi_3, 1)$ and $Q = (\eta_1, \eta_2, \eta_3, 1)$ be any two points on α_3 and α_4 respectively. Then

$$\lambda/\gamma = -l_1(\xi_1, \xi_2, \xi_3)/l_2(\xi_1, \xi_2, \xi_3),$$
$$\nu/\mu = -l_1(\eta_1, \eta_2, \eta_3)/l_2(\eta_1, \eta_2, \eta_3)$$

Hence
$$\frac{\lambda\mu}{\gamma\nu} = \frac{l_1(\xi_1, \xi_2, \xi_3) \quad l_2(\eta_1, \eta_2, \eta_3)}{l_2(\xi_1, \xi_2, \xi_3) \quad l_1(\eta_1, \eta_2, \eta_3)}$$

But as the equations are given in Hessian normal forms, $l_1 (\xi_1, \xi_2, \xi_3)$ is the distance of P from α_1 and similarly for the other functions. Accordingly we may write

$$\frac{\lambda\mu}{\gamma\nu} = \frac{\sin(\alpha_1, \alpha_3)\sin(\alpha_2, \alpha_4)}{\sin(\alpha_2, \alpha_3)\sin(\alpha_1, \alpha_4)}\text{, for planes of a coaxal pencil}$$

$$= \frac{\text{dist}(\alpha_1, \alpha_3)\ \text{dist}(\alpha_2, \alpha_4)}{\text{dist}(\alpha_2, \alpha_3)\ \text{dist}(\alpha_1, \alpha_4)}\text{, for planes of a parallel pencil.}$$

To deduce these formulae it was useful to use Hessian normal forms; for other purposes we may choose the arbitrary factors in a different way. It is however important to notice that the arbitrary factors do not alter the cross-ratios.

Consider now a row of points perspective with a pencil of planes. Given four points of a row

$$P_1 = (a_i), \quad P_2 = (b_i), \quad P_3 = (\gamma a_i + \lambda b_i), \quad P_4 = (\mu a_i + \nu b_i),$$

let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be four planes of a pencil such that P_i is incident with α_i but not with the axis of the pencil ($i = 1, 2, 3, 4$); also let (c_i) and (d_i) be two points incident with the axis of the pencil. The equation of α_1 is then given by

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = 0 \qquad \text{or briefly by } |\, x \quad a \quad c \quad d \,| = 0.$$

It may be remarked that the last three rows of the determinant form a matrix of rank three, so that the equation $l_1 = 0$ of α_1 is not an identity. Similarly the equations of $\alpha_2, \alpha_3, \alpha_4$ are respectively

$$|\, x \quad b \quad c \quad d \,| = 0,$$

$$|\, x \quad \gamma a + \lambda b \quad c \quad d \,| = 0 = \gamma \,|\, x \quad a \quad c \quad d \,| + \lambda \,|\, x \quad b \quad c \quad d \,|,$$

$$|\, x \quad \mu a + \nu b \quad c \quad d \,| = 0 = \mu \,|\, x \quad a \quad c \quad d \,| + \nu \,|\, x \quad b \quad c \quad d \,|$$

Therefore $$(P_1P_2, P_3P_4) = \lambda\mu/\gamma\nu = (\alpha_1\alpha_2, \alpha_3\alpha_4)$$

Thus the cross-ratio of any four points of a row is equal to the cross-ratio of the corresponding four planes of a pencil which is a projection of the row.

Again given four lines g_1, g_2, g_3, g_4 of a pencil of lines in a plane α, let Q be the centre of the pencil, T any point not incident with α and h any line incident with α but not with Q. The cross-ratio of the four planes (Tg_1Tg_2, Tg_3Tg_4) is equal to the cross-ratio of the four points $(hg_1\ hg_2, hg_3\ hg_4)$. This cross-ratio is called the cross-ratio (g_1g_2, g_3g_4) of the four lines of the pencil. We have thus arrived at the property:

The cross-ratio of four elements of a one-dimensional elementary geometric form is not altered by projection or section.

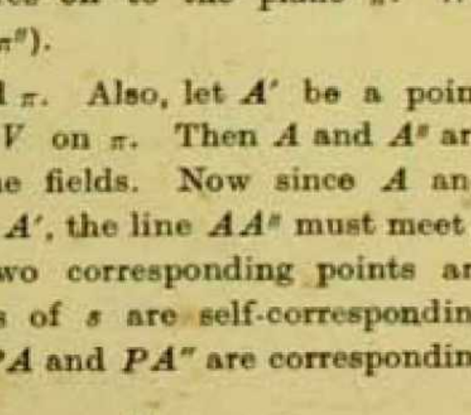
In view of the above considerations we can say that two one-dimensional elementary geometric forms are projective when the elements of the two forms are put in a one-to-one correspondence in such a manner that the cross-ratio of any four elements of one form is equal to the cross-ratio of the corresponding four elements of the other. If, for example, the three points A, B, C of a row are given to correspond respectively to the three planes α, β, γ of a pencil of planes, the point D of the row and the plane δ of the pencil are corresponding elements in this projectivity if

$$(AB, CD) = (\alpha\beta, \gamma\delta)$$

What have been said above regarding projectivity between two one-dimensional geometric forms hold also for two-dimensional geometric forms when to each one-dimensional form in one corresponds a projective one-dimensional form in the other.

55.1. Homology. Let π and π' be two planes intersecting in a line s. Take a line g skew to s and on this line take two points U and V. Project the plane field (π') from U, V as centres on to the plane π. We thus obtain on π two cobasal plane fields (π), (π'').

Let S be the point of intersection of g and π. Also, let A' be a point of π' and A, A'' its projections from U, V on π. Then A and A'' are corresponding points of the two cobasal plane fields. Now since A and A'' both lie on the plane through g and A', the line AA'' must meet g and therefore pass through S. Therefore two corresponding points are collinear with S. The point S and all points of s are self-corresponding points. So if P is a point of s, the lines PA and PA'' are corresponding lines. Therefore, two corresponding lines intersect on s and every line in π through S is a self-corresponding line. Thus we arrive at the following property :

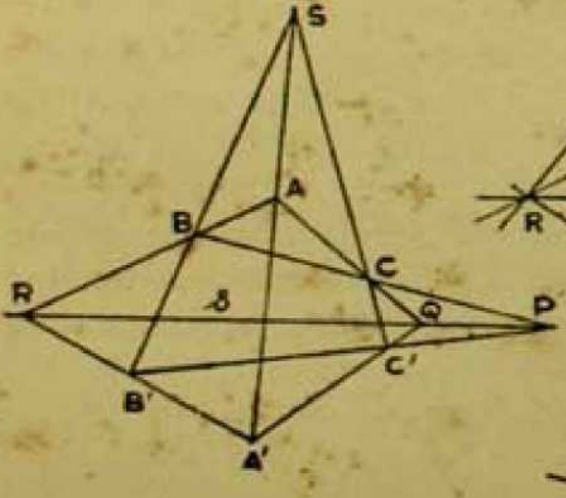


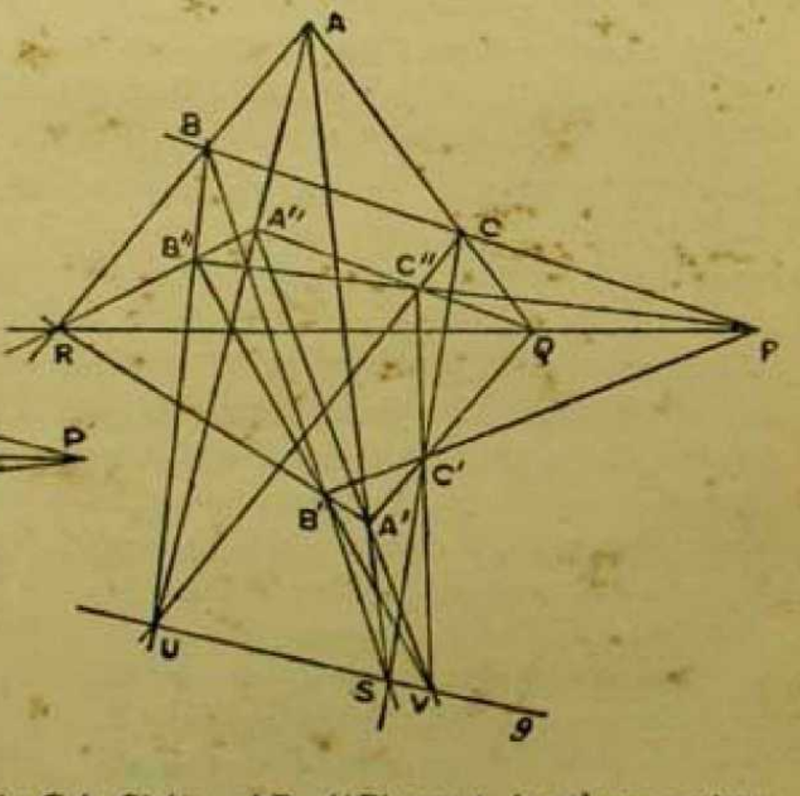
Two cobasal plane fields which are the projections of one and the same plane field from two different centres are such that the join of corresponding points passes through a fixed point S and the meet of corresponding lines lies on a fixed line s.

Two plane fields, whether cobasal or not, which have the property that the join of corresponding points are concurrent in point S and the meet of corresponding lines are collinear with line s are said to be *homologous* or *perspective*; S is called the *centre* and s the *axis* of homology or of perspectivity.

DESARGUES' theorem on perspective triangles. If two triangles ABC and $A'B'C'$ are such that the lines AA', BB', CC' joining the three pairs of vertices are concurrent, then the three pairs of corresponding sides BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ intersect in three collinear points, and conversely.

Two cases arise according as the two triangles lie in two different planes or in the same plane. First suppose that the triangles lie in different planes. Let AA', BB', CC' meet in a point S. Since BB', CC' are coplanar, BC, $B'C'$ are also coplanar and so meet in a point P. Similarly, CA, $C'A'$ meet in a point Q and AB, $A'B'$ meet in a point R. Now, P lies in the plane ABC and also in the plane $A'B'C'$; therefore P lies on the line of intersection s of these two planes. Similarly, Q and R lie on s.





Conversely, if BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ meet in three points, they must meet on the line of intersection of the planes ABC and $A'B'C'$. Therefore BB', CC' are coplanar and so meet in a point; similarly, CC', AA' meet in a point and AA', BB' also meet in a point. But as the three lines AA', BB', CC' are not coplanar, they must be concurrent.

There are ten points and ten lines in this configuration which are the points and lines of intersection of five planes, three through S and two through s.

Secondly suppose that the two triangles lie in a plane π and that AA', BB', CC' are concurrent in S. As in the case of two homologous figures, take a line g passing through S but not lying in π, and on this line take two points U, V. Since UV, AA' meet in S, they are coplanar and so UA, VA' meet in a point A''. Similarly, UB, VB' meet in B'' and UC, VC' meet in C''. The two triangles ABC, $A''B''C''$ are now such that they lie on two different planes π, π'' and that the lines AA'', BB'', CC'' meet in U. Therefore, by the previous case just proved, the three pairs of corresponding sides meet in three collinear points on the line of intersection s of π, π''. But the same thing is also true of the triangles $A'B'C'$, $A''B''C''$, as may be seen by interchanging U and V; and the points of intersection of the corresponding sides of these two triangles must be the same as before, namely the points where the sides of the triangle $A''B''C''$ meet s. Hence, BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ meet in three collinear points.

Conversely, let BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ meet in three points on a line s and AA', BB' meet in S. Then, if SC did not pass through C', it would cut $B'C'$ in some other point D'. It would follow, by what we have just proved, that $A'D'$, AC meet s in the same point. But this is impossible unless D' coincides with C'.

A proof for the case of two coplanar triangles has already been given in § 36, but the proof given here is purely projective.

CHAPTER XIV

GROUPS OF TRANSFORMATIONS AND CLASSIFICATION OF GEOMETRIES

56. Dimensionality of spaces. The points of a Euclidean space are represented in a continuous manner by 3 parameters, the points of a Euclidean plane by two parameters and the points of a Euclidean straight line by one parameter only. For this reason these mathematical entities are said to be spaces of three, two and one dimensions respectively. The projective space, the projective plane and the projective straight line need one parameter more than the corresponding Euclidean entities ; but as the parameters have a common arbitrary factor, we call them three-dimensional, two-dimensional and one-dimensional projective spaces.

Correspondingly, a pencil (of planes, lines, circles, etc.) can be considered as a one-dimensional and a bundle as a two-dimensional "space". Further, the conics of a projective plane depend on six parameters ; to every set of six values, except all zero, there corresponds a conic ; but again as there is a common arbitrary factor, these conics form a projective "space" of five dimensions. The "points" of this space are conics, the "rows of points" are pencils of conics etc. ; therefore theorems on rows of points can be applied to pencils of conics.

Many geometrical considerations may be treated simultaneously by introducing the more general notion of an *n-dimensional space.* The reader may have realised that there is a close resemblance between the geometry of the plane and that of the space. Indeed most of the methods used, as well as a portion of the results, are independent of the number n of dimensions. The n-dimensional projective space is composed of points

$$\rho\,(x_1, \ldots x_{n+1}),\ \text{where}\ (x_1, \ldots, x_{n+1}) \neq (0, \ldots, 0), \rho \neq 0 \qquad (14.1)$$

Thus for $n = 1, 2, 3,$ we get the projective line, the projective plane, and the projective (3-dimensional) space respectively. If $n > 1$, we can easily find inside an n-dimensional space subspaces of lower dimensions. For this purpose, consider a homogeneous system of linear equations in the coordinates of an n-dimensional projective space, the rank of the matrix of the coefficients being r :

$$\begin{aligned} a_1x_1 + \ldots\ldots + a_{n+1}x_{n+1} &= 0 \\ b_1x_1 + \ldots\ldots + b_{n+1}x_{n+1} &= 0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ k_1x_1 + \ldots\ldots + k_{n+1}x_{n+1} &= 0 \end{aligned} \qquad (14.2)$$

All solutions of such a system depend on $n-r+1$ independent solutions (the solutions are '$(n+1)$-vectors' in the sense of algebra), say $\alpha_1, \ldots, \alpha_{n-r+1}$. Therefore every solution is given uniquely by

$$t_1\alpha_1 + \ldots + t_{n-r+1}\alpha_{n-r+1}, \qquad (14.3)$$

where the t's are arbitrary numbers. Consider the solutions for which

$$(t_1, \ldots, t_{n-r+1}) \neq (0, \ldots, 0) \qquad (14.4)$$

A common factor of (14.4) furnishes a common factor of (14.3), and so these solutions correspond to those points (14.1) which satisfy the equations (14.2). As the t's are $n-r+1$ parameters, these points form a projective $(n-r)$-dimensional space, which is a subspace of the given n-dimensional projective space

E.g., for $n = 3$, $r = 1$, we have $n-r = 2$. The points satisfying

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$$

therefore form a projective plane. For $n = 3$, $r = 2$, we have $n-r = 1$. The points satisfying two independent linear equations

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 &= 0 \\ b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 &= 0 \end{aligned}$$

therefore form a projective straight line, namely the axis of the pencil generated by the two planes which correspond to the two linear equations.

Every linear equation, *e.g.* of (14.2), is called a "hyperplane" or a "prime". As in § 53, an n-dimensional projective space (14.1) can be made up of a hyperplane at infinity

$$V \equiv \rho(x_1, \ldots, x_n, 0) \qquad (14.5)$$

and an n-dimensional Euclidean space

$$S_n \equiv (x_1, \ldots, x_n) \qquad (14.6)$$

which is identical with the set of points $\rho(x_1, \ldots, x_n, 1)$. S_n is, of course, uniquely defined by the n *non-homogeneous* parameters $x_1, \ldots, x_n$. On the other hand, an n-dimensional Euclidean space can be represented by (14.6) and can be extended to a projective space by adding, so to say, the elements at infinity (14.5). By this extension, the formulae of projective geometry become simpler and they show a duality which does not hold in Euclidean geometry.

57. Groups of transformations. Let us consider linear transformations in $n+1$ homogeneous coordinates or in n non-homogeneous coordinates. A set of linear transformations $T_1, T_2, T_3, \ldots$, which may be finite or infinite in number, is said to form a *group* if the transformations of the set obey the following two laws :

(1) The product (or resultant) of every pair of transformations (including the product of a transformation by itself) is a transformation which belongs to the set.

(2) Every transformation of the set has its inverse and the inverse belongs to the set. If the inverse of a transformation T_r is denoted by T_r^{-1}, the inverse has the following property :

$$T_r T_r^{-1} = T_r^{-1} T_r = I,$$

the identical transformation or the identity.

It follows from the definition that if the linear transformations T_i of a set form a group, $T_p T_q = T_k$ is a transformation of the group. Therefore $T_r T_p T_q = T_r T_k = T_l$, a transformation of the group. In general, the product of any number of transformations is a transformation of the group. Also, the identical transformation I is a transformation of the group.

The product $T_p T_q$, which is taken to mean that T_q is followed by T_p, is not necessarily equal to the product $T_q T_p$. A group of linear transformations is said to be *commutative* if, for every pair of transformations of the group, $T_q T_p = T_q T_p$; otherwise the group is *non-commutative*. Further, since the T_i's are linear transformations, $T_p (T_q T_r) = (T_p T_q) T_r$, showing that the product is *associative*.

Now suppose that we are given a group of linear transformations. We may then obtain a set of particular transformations of the given group by imposing one or more conditions on each transformation of the group. If this set of particular transformations obeys the above two laws of a group, it is said to form a *subgroup* of the given group.

All linear transformations of the type

$$\rho x_i' = \sum_{j=1}^{n+1} a_{ij} x_j, \quad | a_{ij} | \neq 0, \quad i = 1, 2, \ldots, n+1 \qquad (14.7)$$

are called projective transformations or *collineations* of the n-dimensional projective space. If a collineation

$$\rho\,\bar{x}_j = \sum_{k=1}^{n+1} b_{jk}\, x_k \qquad | b_{jk} | \neq 0$$

is followed by a collineation

$$\sigma\, x_i' = \sum_{j=1}^{n+1} c_{ij}\, \bar{x}_j', \qquad | c_{ij} | \neq 0$$

the product is the transformation

$$\varpi\, x_i' = \sum_{j,k=1}^{n+1} c_{ij}\, b_{jk}\, x_k, \qquad i = 1, 2, \ldots, n+1 \tag{14.7'}$$

Since

$$| \sum_j c_{ij}\, b_{jk} | = | c_{ij} | \;\; | b_{jk} |$$

is different from zero, the product (14.7′) is a collineation.

Again the inverse of (14.7) is the transformation

$$\rho''\, x_i = \sum_{j=1}^{n+1} A_{ji}\, x_j', \qquad i = 1, 2, \ldots, n+1,$$

where A_{ij} are the cofactors of a_{ij} in the determinant $| a_{ij} |$.

Since

$$| a_{ij} | \neq 0, \quad \text{so} \quad | A_{ij} | \neq 0$$

Hence the inverse is also a collineation. Therefore *the set of all collineations of an n-dimensional projective space form a group, the identical transformation of this group being*

$$\rho\, x_i' = x_i, \qquad i = 1, 2, \ldots, n+1$$

When $n = 2$, all collineations of a plane form a group ; when $n = 3$ all collineations of a 3-dimensional space form a group, and so on.

Now let us impose the following condition on the collineations (14.7) :

$$a_{n+11} = a_{n+12} = \ldots\ldots = a_{n+1n} = 0$$

The transformations reduce to

$$\begin{aligned} \rho\, x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n+1}x_{n+1} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \rho\, x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn+1}x_{n+1} \\ \rho\, x_{n+1}' &= \qquad\qquad\qquad\qquad a_{n+1n+1}x_{n+1} \end{aligned} \tag{14.8}$$

with

$$a_{n+1n+1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0 \tag{14.8'}$$

All transformations (14.8) with (14.8′) are called the *affinities* of the n-dimensional space. In an affinity, therefore, a special $(n-1)$-dimensional subspace, namely $x_{n+1} = 0$, is kept fixed. When $n > 3$, this special subspace is called the *hyperplane at infinity*; when $n = 2, 3$ we have the line at infinity in a plane, the plane at infinity in a 3-dimensional space respectively which are kept fixed by affinity.

It can be shown, as in the case of collineations, that the product of two affinities is an affinity and that the inverse of an affinity is an affinity. Therefore the set of all affinities of an n-dimensional space form a group. Thus, when $n = 2$, all affinities of a plane form a group; when $n = 3$, all affinities of a 3-dimensional space form a group, and so on. Further, since affinities are obtained from collineations by imposing a particular condition, *the group of affinities is a subgroup of the group of collineations.*

Again, since $a_{n+1\,n+1} \neq 0$, we can, without loss of generality, take $a_{n+1\,n+1} = 1$. The equations of affinities can then be written in non-homogenous coordinates as

$$\begin{aligned} x' &= a_{11}x + a_{12}y + \dots + a_{1n}w + a_{1\,n+1} \\ y' &= a_{21}x + a_{22}y + \dots + a_{2n}w + a_{2\,n+1} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ w' &= a_{n1}x + a_{n2}y + \dots + a_{nn}w + a_{n\,n+1} \end{aligned} \qquad (14.9)$$

with

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{vmatrix} \neq 0$$

Now suppose we impose the following conditions on the affinities (14.9):

$$\sum_{k=1}^{n} a_{ik}\,a_{jk} = \begin{bmatrix} c^2, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{bmatrix} \qquad i, j = 1, 2, \dots, n. \qquad (14.9')$$

where c is a constant $\neq 0$. Evidently there are n conditions when $i = j$ and $n(n-1)/2$ when $i \neq j$. It follows

$$\Delta^2 = c^{2n}, \text{ so } \Delta = \pm c^n$$

All transformations (14.9) with (14.9′) are called *similarities* of n-dimensional space, and all similarities form a group. When $\Delta > 0$, the similarities are said to be *direct*. It can be seen that *the group of similarities is a subgroup of the group of affinities.*

Further, if on the group of similarities we impose the condition $c^2 = 1$, we obtain the transformations (14.9) with the conditions

$$\sum_{k=1}^{n} a_{ik}\, a_{jk} = \left| \begin{array}{l} 1, \text{ if } i = j \\ \\ 0, \text{ if } i \neq j \end{array} \right. \qquad i, j = 1, 2, \ldots, n, \quad \text{so} \quad \Delta^2 = 1 \quad (14.9'')$$

Transformations (14.9) satisfying (14.9″) form a *group of orthogonal transformations.*

All transformations (14.9) with the condition (14.9″) for which $\Delta = +1$ are called *rigid motions* of the n-dimensional space and all the rigid motions form a group. Evidently *the group of rigid motions is a subgroup of the group of similarities and therefore of the group of affinities.* The group of rigid motions of a plane can be written in the form (3.1), as has already been noticed, and they form a group. Similarly, all rigid motions of a three-dimensional space form a group.

All transformations (14.9) with (14.9″) for which $\Delta = -1$ are called *symmetries* of the n-dimensional space. Since

$$\Delta^{2m} = +1, \quad \Delta^{2m-1} = -1,$$

where m is a positive integer, the product of an even number of symmetries is a rigid motion and the product of an odd number is a symmetry. Therefore the symmetries do not form a group. The equations of symmetries of a plane can be written in the form (5.2), as has already been noticed.

Consider the transformations obtained from the group of similarities by imposing the conditions

$$a_{ij} = \left[\begin{array}{l} c, \text{ if } i = j \\ \\ 0, \text{ if } i \neq j \end{array} \right. \qquad i, j = 1\,; 2, \ldots, n$$

These transformations can therefore be written as

$$\begin{aligned} x' &= cx + c_1 \\ y' &= cy + c_2 \qquad\qquad c \neq 0 \qquad\qquad (14.10) \\ &\cdots\cdots\cdots \\ w' &= cw + c_n \end{aligned}$$

All transformations (14.10) are called *homothetic transformations* of the n-dimensional space and they *form a group.*

Finally, consider the transformations obtained from the group of rigid motions by imposing the condition

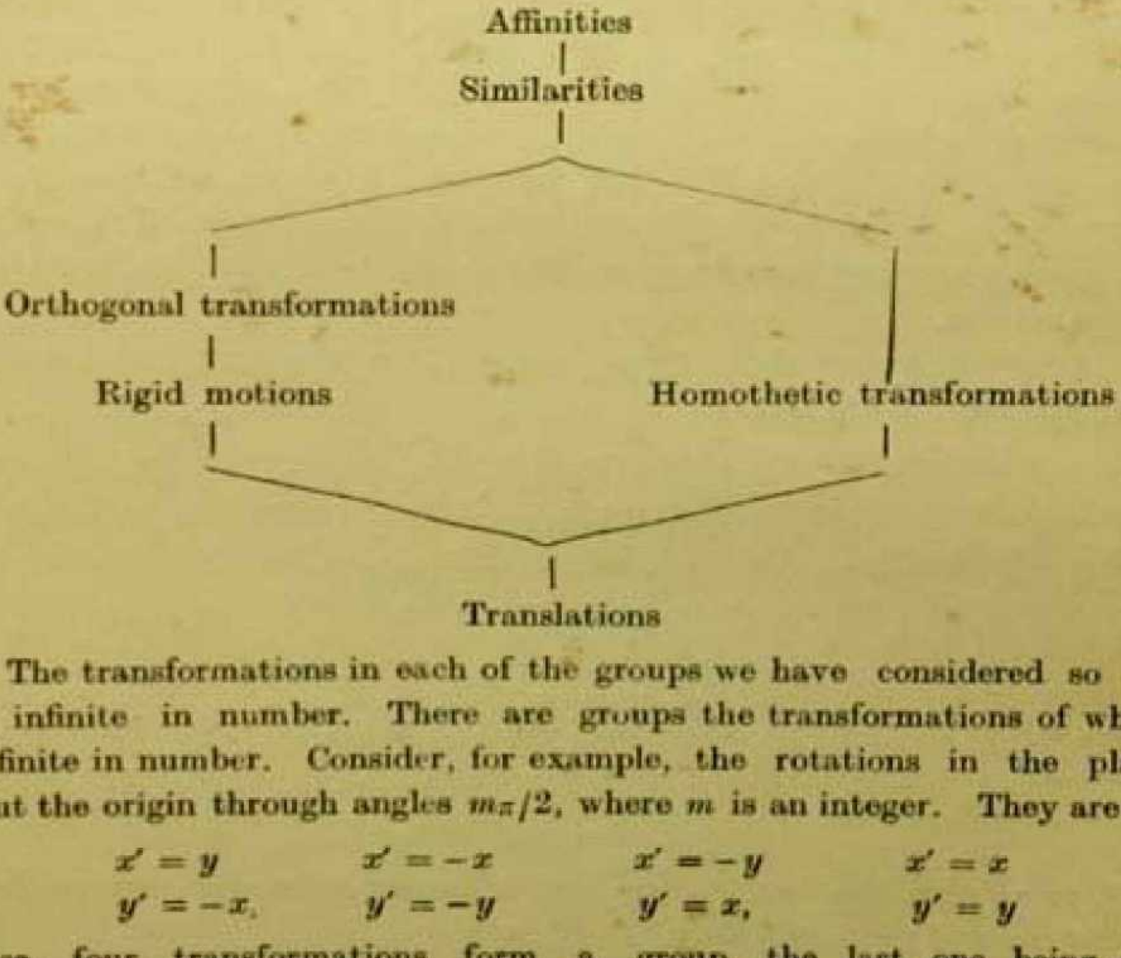
$$a_{ij} = \left[\begin{array}{l} 1, \text{ if } i = j \\ \\ 0, \text{ if } i \neq j \end{array} \right. \qquad i, j = 1, 2, \ldots n$$

The transformations thus obtained can be written as

$$\begin{aligned} x' &= x + c_1 \\ y' &= y + c_2 \\ &\cdots\cdots \\ w' &= w + c_n \end{aligned} \tag{14.11}$$

All transformations (14.11) are called *translations* of the n-dimensional space, and *the translations form a group.* Since this group can also be obtained from the group of transformations (14.10), the group of translations is a subgroup of both the groups of rigid motions and homothetic transformations and therefore of the groups of similarities and the affinities. The group (14.11) is a commutative group.

We may exhibit the different subgroups of the affine group as follows :



The transformations in each of the groups we have considered so far are infinite in number. There are groups the transformations of which are finite in number. Consider, for example, the rotations in the plane about the origin through angles $m\pi/2$, where m is an integer. They are

$$\begin{matrix} x' = y & \quad x' = -x & \quad x' = -y & \quad x' = x \\ y' = -x, & \quad y' = -y & \quad y' = x, & \quad y' = y \end{matrix}$$

These four transformations form a group, the last one being the identical transformation. Similarly, the following two rotations in the plane about the origin through $m\pi$, m being an integer, from a group :

$$\begin{matrix} x' = -x & \quad x' = x \\ y' = -y & \quad y' = y \end{matrix}$$

58. Classification of geometries. Geometry deals mainly with properties of figures. It was pointed out by *FELIX KLEIN* (1849-1925) that properties of figures may be classified according to the manner in which they behave when subjected to certain transformations.

A property which remains invariant under the group of collineations is called a *projective property* and the geometry which deals with projective properties is called the *projective geometry.* We have enumerated up till now various properties of figures which behave in this manner ; *e.g.*, collinearity of points and concurrency of lines, cross-ratio, pole and polar, the principle of duality, etc., are projective properties. Properties which remain invariant by every projection or section are therefore projective properties. Since distance, angle, area, volume do not remain invariant under the group of collineations, there is no place of these notions in projective geometry. A theorem which deals merely with projective properties is a projective theorem, *e.g.*, Desargues' theorem on perspective triangles.

A property which remains invariant under the group of affine transformations, but not under the projective group, is called an *affine property.* The geometry which deals with affine properties, either in themselves or in conjunction with projective properties, is called the *affine geometry.* Since the affine group is a subgroup of the projective group, any property which remains invariant under the projective group also remains invariant under the affine group, but the converse is not true ; *e.g.*, the property of cross-ratio and, in particular, the harmonic properties of complete quadrangle and quadrilateral are projective properties ; these properties are also invariant under the affine group. But parallelism of lines and the ratio of algebraic distances are affine properties ; these properties do not remain invariant under the projective group. We have enumerated in the plane geometry various properties which are affine, *e.g.*, central properties of conics, property of conjugacy of diameters of conics, etc. The theorem that the cross-ratio of four diameters of a conic is equal to the cross-ratio of the four conjugate diameters combines both projective and affine properties and is therefore a theorem of affine geometry.

A property which remains invariant under the group of rigid motions, but not under the affine group, is called a *metric property.* The study of these properties, either in themselves or in conjunction with affine and projective properties, constitutes the *metric geometry.* Since the metric group is a subgroup of the affine group, a property which holds in affine geometry also holds in metric geometry, but not *vice versa.* We have seen in §§ 10.1, 23.2 that distance, angle, area remain invariant under the group of rigid motions but not under the affine group. So, these notions are metric. The ordinary Euclidean geometry is a metric geometry and the Pythagorean theorem is a metric theorem. Besides the Euclidean geometry, there are other geometries which are metric, *e.g.*, the so-called non-Euclidean geometry.

The projective, the affine and the metric are three of the more important classes of geometries.

Two figures are said to be *equivalent* under a particular group of transformations if there exists a transformation of the group which carries one figure into the other. We have seen in § 24 that any two parabolas or any two ellipses or any two hyperbolas are equivalent under the affine group and in § 39 that any two nondegenerate conics (with real traces) are equivalent under the projective group. This is expressed by saying that all parabolas or all ellipses or all hyperbolas are equivalent in affine geometry and all (real) nondegenerate conics are equivalent in projective geometry.

Although, as has been seen in §§ 32, 33, 52, the introduction of homogeneous coordinates is necessary for the study of (analytical) projective geometry, the affine and the metric geometries are in themselves geometries of ordinary space; that is to say, it is not necessary to use homogeneous coordinates in these geometries. In both affine and metric geometries the principle of duality fails to function.

We have noticed that the affine geometry is obtained from the projective geometry by certain specialisation and that the metric geometry is derived from the affine geometry also by specialisation. In order to understand the nature of these specialisations we need homogeneous coordinates. To illustrate the implications consider the case of the plane geometry. We have seen in § 43 that the affine plane geometry is derived from the projective plane geometry by keeping one of the lines, called the line at infinity, fixed. Again, it would follow from § 44 that the group of rigid motions of the plane in homogeneous coordinates is given by

$$\begin{aligned} \rho x_1' &= ax_1 + bx_2 + c_1x_3 \\ \rho x_2' &= -bx_1 + ax_2 + c_2x_3 \qquad a^2 + b^2 = 1 \\ \rho x_3' &= x_3 \end{aligned}$$

and that the circular points at infinity I and J have the coordinates $(i, 1, 0)$, $(-i, 1, 0)$, $i^2 = -1$. The coordinates of the point into which $(i, 1, 0)$ is transformed by the above group are :

$$\rho x_1' = ai + b = i\,(a - bi)$$
$$\rho x_2' = -bi + a = (a - bi)$$
$$\rho x_3' = 0 \qquad = 0\,(a - bi)$$

This shows that the point $(i, 1\ 0)$ remains fixed. Similarly for the point $(-i, 1, 0)$. Thus, the group of rigid motions not only preserves distance, transforms the line at infinity into itself but leaves the two circular points fixed. It may be noted that under the group of similarity transformations the circular points are left fixed but distance is not preserved.

Moreover, it has been seen in § 14.1 that if l_1 and l_2 are two lines through a point, and l_I, l_J are the two isotropic lines through the same point, the angle θ between l_1 and l_2 is defined by

$$\theta = \frac{1}{2i} \log\,(l_1\, l_2,\, l_I\, l_J)$$

This is the projective way of introducing angle in metric geometry. That the angle is a metric notion may also be seen as follows : The group of rigid motions preserves cross-ratio and transforms an isotropic line into an isotropic line of the same kind. Therefore if l_1', l_2', l_I', l_J' are respectively the transforms of l_1, l_2, l_I, l_J by an arbitrary rigid motion, we have

$$(l_1\, l_2,\ \ l_I\, l_J) = (l_1'\, l_2',\ \ l_I'\, l_J')$$

Hence the angle is an invariant under the group of rigid motions.

As we are mainly concerned with geometries of dimensions not greater than three, we shall henceforth use the word 'space' to mean a space of three-dimensions unless otherwise stated.

CHAPTER XV

PROJECTIVE THEORY OF QUADRICS

59. Projective properties of quadrics. The general equation of the second degree in homogeneous point coordinates can be written as

$$\sum_{i,j} a_{ij} x_i x_j = 0 \quad i, j = 1, 2, 3, 4, \quad a_{ij} = a_{ji} \qquad (15.1)$$

where the coefficients a_{ij} are not all zero. All surfaces represented by (15.1) are surfaces of the second degree, generally known as *quadrics*. The determinant $| a_{ij} |$ is called the discriminant of the equation (15.1).

A point (r_i) is called a *singular point* of the quadric (15.1) if and only if

$$\sum_{i,j=1}^{4} a_{ij} r_i x_j \equiv 0$$

for all points (x_i) of the space. When the quadric has a singular point, we call the quadric a *singular quadric*. It follows that the necessary and sufficient condition that a quadric be singular is that $| a_{ij} | = 0$, (See § 42).

Consider a point $(\gamma r_i + \lambda s_i)$ on the line joining two points (r_i) and (s_i). The points of intersection of the line and the quadric (15.1) are determined by the roots of the equation

$$\gamma^2 \sum a_{ij} r_i r_j + 2\gamma\lambda \sum a_{ij} r_i s_j + \lambda^2 \sum a_{ij} s_i s_j = 0 \qquad (15.2)$$

So, a line may intersect the quadric in two points, either distinct or coincident, may not intersect the quadric at all or may lie wholly on the quadric.

A *tangent line* is defined as a line which meets the quadric in two ultimately coincident points or lies wholly on the quadric. This line is said to be a tangent to the quadric at each point which it has in common with the quadric.

Let the quadric be nonsingular, *i.e.*, $| a_{ij} | \neq 0$. The discriminant of (15.2) is

$$(\sum a_{ij} r_i s_j)^2 - (\sum a_{ij} r_i r_j)(\sum a_{ij} s_i s_j)$$

If (r_i) is a given point of the quadric and the two roots of (15.2) are equal, we must have $\Sigma\, a_{ij} r_i s_j = 0$. Or, expressing in current coordinates,

$$\sum a_{ij} r_i x_j = 0 \qquad (15.3)$$

This shows that the tangent lines to the quadric at a given point (r_i) form a pencil of lines. The equation (15.3) represents the plane of the pencil and this plane is called the *tangent plane* to the quadric at the point (r_i).

Let neither of the points (r_i), (s_i) be a point of the quadric. The condition that the sum of the roots of (15.2) is zero is $\Sigma a_{ij} r_i s_j = 0$. If now we suppose that the line joining (r_i) and (s_i) meets the quadric, the condition that the two points (r_i) and (s_i) are separated harmonically by the points of intersection of the line and the quadric is $\Sigma a_{ij} r_i s_j = 0$. If (r_i) is given, all points which are harmonically separated from (r_i) by the quadric lies in the plane

$$\sum a_i r_i x_j = 0$$

This plane is called the *polar plane* of the point (r_i) with respect to the quadric. The point (r_i) is called the *pole* of the polar. Comparing this equation of the polar plane with the equation (15.3) of the tangent plane, we *define* the polar of any point (r_i) with respect to the quadric (15.1) as the plane given by the equation (15.3), whether the point (r_i) lies on the quadric or not. If a point (s_i) lies on the polar of a point (r_i), then

$$\sum a_{ij} r_i s_j = 0 \tag{15.4}$$

This shows that the point (r_i) also lies on the polar of (s_i). Two such points (r_i), (s_i) satisfying the relation (15.4), as well as their polars, are said to be *conjugate* to one another.

As in the plane geometry where we have polar (or self-polar or self-conjugate) triangles with respect to a nondegenerate conic, any two of the vertices or sides of such a triangle being conjugate to one another, so in the space geometry we have *polar* (or *self-polar* or *self-conjugate*) *tetrahedrons* with respect to a nonsingular quadric, any two of the vertices or faces of such a tetrahedron being conjugate to one another.

Let the polar of the two points (r_i) and (s_i) with respect to (15.1) be respectively (u_i) and (v_i) in plane coordinates. So, considering the coefficient of (15.3),

$$\rho u_i = \sum_k a_{ik} r_k, \quad \sigma r_i = \sum_k c_{ik} s_k$$

It follows that the polar of any point $(\gamma r_i + \lambda s_i)$ on the line p joining the two points is $(\gamma u_i + \lambda v_i)$ and therefore always passes through the line of intersection g of the planes (u_i) and (v_i). Thus, the polars of a row of points form a pencil of planes. If (t_i) be any point on the line g, then for all values of γ and λ

$$\sum_i (\gamma u_i + \lambda v_i) t_i = 0 \; ; \quad \text{so} \quad \sum_{i,\,k} a_{ik}(\gamma r_k + \lambda s_k) t_i = 0$$

This shows that every point of g is conjugate to every point of p. Also, if (w_i) be the polar of (t_i), it follows from above that, for all values of γ and λ,

$$\sum_k (\gamma r_k + \lambda s_k)\, w_k = 0$$

This shows that the polar of every point of g passes through p.

Thus *two lines, each of which is the line of intersection of the polars of points of the other, are such that two points, one on each, are always conjugate.* Two such lines are called *polar lines.*

Two lines each of which intersects the polar line of the other are called *conjugate lines.* In particular, a line is self-conjugate when it intersects its own polar. Two polar lines constitute a special case of two conjugate lines. In a polar tetrahedron, every pair of vertices, every pair of faces and every pair of edges are conjugate and every pair of opposite edges are polar lines.

The self-conjugate points, the self-conjugate planes, the self-conjugate lines and the self-polar lines with respect to a nonsingular quadric are respectively the points, the tangent planes, the tangent lines of the quadric and the lines lying wholly on the surface (called, *generators*), if any, of the quadric.

If the coordinates are so chosen that the four points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) are the vertices of a polar tetrahedron, then, since (15.4) is satisfied for two conjugate points, the equation of the quadric reduces to the form

$$\sum a_{ii} x_i^2 = 0 \tag{15.5}$$

In other words, (15.5) is the equation of a quadric referred to a polar tetrahedron as the *tetrahedron of reference.*

60. Collineation. Consider a collineation of the space in point coordinates

$$\rho x'_i = \sum_k a_{ik} x_k, \quad i = 1, 2, 3, 4, \quad |\, a_{ik} \,| \neq 0 \tag{15.6}$$

Its inverse is given by

$$\rho\, x_i = \sum_k A_{ki}\, x_k', \quad i = 1, 2, 3, 4, \tag{15.6'}$$

where A_{ik} are the cofactors of a_{ik} in $|a_{ik}|$. Let $\Sigma u_i x_i = 0$ be a plane and let this plane be transformed into the plane $\Sigma u'_i x'_i = 0$ by (15.6′). So

$$0 = \sum_i u_i x_i \quad = \sum_{i,k} u_i A_{ki} x'_k \quad = \sum_k u'_k x'_k$$

Therefore $$\sigma' u'_k = \sum_i A_{ki} u_i, \qquad k = 1, 2, 3, 4 \tag{15.7}$$

Also

$$0 = \sum_i u'_i x'_i \quad = \sum_{i,k} u'_i a_{ik} x_k \quad = \sum_k u_k x_k$$

Therefore $$\sigma u_k = \sum_i a_{ik} u'_i, \quad k = 1, 2, 3, 4 \tag{15.7′}$$

Thus, (15.7) is the collineation in plane coordinates and (15.7′) its inverse. Evidently the transformations (15.6), (15.6′), (15.7), (15.7′) represent the same collineation.

As in (9.4), let r points $(x_{1i}), (x_{2i}), \ldots, (x_{ri})$ be *linearly dependent*, so that

$$\sum_{t=1}^{r} \lambda_t x_{ti} = 0, \qquad i = 1, 2, 3, 4,$$

$\lambda_1, \lambda_2, \ldots \lambda_r$ being r constants not all zero. Let these r points be transformed into the r points $(x'_{1i}), \ldots, (x'_{ri})$ respectively by the collineation (15.6). So

$$\sum_{k=1}^{4} \sum_{t=1}^{r} \lambda_t A_{ki} x'_{tk} = 0, \qquad i = 1, 2, 3, 4$$

Multiply these four relations by $a_{\nu 1}, a_{\nu 2}, a_{\nu 3}, a_{\nu 4}$ respectively and add. Then

$$0 = \sum_{i,k=1}^{4} \sum_{t=1}^{r} \lambda_t A_{ki} a_{\nu i} x'_{tk} \quad = \sum_{i,k=1}^{4} A_{ki} a_{\nu i} \sum_{t=1}^{r} \lambda_t x'_{tk} = |a_{ik}| \sum_{t=1}^{r} \lambda_t x'_{t\nu}$$

As $|a_{ik}| \neq 0$,

$$\sum_{t=1}^{r} \lambda_t x'_{t\nu} = 0, \qquad \nu = 1, 2, 3, 4$$

Therefore the r transformed points are also linearly dependent. Thus, *linearly dependent points are transformed by collineation into linearly dependent points.* It follows from above that if

$$x_{3i} = \mu x_{1i} + \nu x_{2i}, \quad x_{4i} = \mu' x_{1i} + \nu' x_{2i},$$

so that the four points $(x_{1i}), (x_{2i}), (x_{3i}), (x_{4i})$ are collinear, then

$$x'_{3i} = \mu x'_{1i} + \nu x'_{2i}, \quad x'_{4i} = \mu' x'_{1i} + \nu' x'_{2i}$$

Hence *the cross-ratio of four collinear points remains unaltered by collineation.*

Also, analogous to the theorem in § 34.1, we have, in the projective space, the following theorem.

Theorem. There exits a unique collineation which transforms five given points, no four of which are coplanar, into five other given points, no four of which are also coplanar.

Proof : As the five given points $(x_{1i}), (x_{2i}), \ldots, (x_{5i})$ are such that no four of them are coplanar, the matrix

$$\begin{pmatrix} x_{11} & x_{21} & x_{31} & x_{41} & x_{51} \\ x_{12} & x_{22} & x_{32} & x_{42} & x_{52} \\ x_{13} & x_{23} & x_{33} & x_{43} & x_{53} \\ x_{14} & x_{24} & x_{34} & x_{44} & x_{54} \end{pmatrix}$$

has the property that no fourth order determinant formed out of any four columns can vanish. Let these five points be transformed by a collineation (15.6) (as yet unknown) into five other points which we can suppose, for the sake of simplicity, to be the fundamantal points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 1) respectively. Apply (15.6′). We shall have to take five different arbitrary constants ρ for the five different pairs of points. For the first pair of points

$$\rho_1 x_{1i} = \sum A_{1i} x'_k = A_{1i}$$

Thus, for the first four pairs of points,

$$A_{ki} = \rho_k x_{ki}, \qquad k, i = 1, 2, 3, 4 \tag{15.8}$$

For the fifth pair of points,

$$\rho_5 x_{5i} = \sum_{k=1}^{4} A_{ki}.$$

Substituting for A_{ki},

$$\rho_5 x_{5i} = \sum \rho_k x_{ki}, \qquad i = 1, 2, 3, 4$$

These are four linear homogeneous equations in the ρ's. Since the x's are given points satisfying the given condition regarding the rank of the matrix formed by their coordinates, the ratios $\rho_1/\rho_5, \rho_2/\rho_5, \rho_3/\rho_5, \rho_4/\rho_5$ are determined none of which can be zero ; that is, the ratios $\rho_1 : \rho_2 : \rho_3 : \rho_4$ are known. Therefore, by (15.8), the coefficients A_{ki} are known, except for an arbitrary common factor. Hence (15.6′) and so (15.6) is known. Thus the collineation is uniquely determined.

61. Projective classification of quadrics. We start with the general equation of a quadric in homogeneous coordinates, namely

$$\sum_{i,j} a_{ij} x_i x_j = 0, \quad i, j = 1, 2, 3, 4, \quad a_{ij} = a_{ji} \tag{15.9}$$

where the coefficients a_{ij} are not all zero.

We notice first that, without loss of generality, we may assume that *at least one* of the quantities $a_{11}, a_{22}, a_{33}, a_{44}$ is other than zero. For, suppose that all these four quantities are zero ; then there is at least one of the quantities a_{ij}, $i \neq j$, which is not zero. For the sake of definiteness, let $a_{1k} \neq 0$ when k has a definite value out of 2, 3, 4. The general equation (15.9) can then be written as

$$\sum a_{ij} x_i x_j + 2x_k \sum a_{ik} x_i = 0, \quad i \neq j \neq k$$

Apply the collineation

$$x_k = x'_1 + x'_k, \quad x_j = x'_j, \quad \text{for all } j \neq k$$

The equation reduces to

$$\sum a_{ij} x'_i x'_j + 2(x'_1 + x'_k) \sum a_{ik} x'_i = 0, \quad i \neq j \neq k,$$

or to the form

$$2\, a_{1k} x'^2_1 + \sum b_{ij} x'_i x'_j = 0, \quad i \neq j \, ;$$

in the summation i, j take up all the values $1, \ldots, 4$. Hence, we see that it has been possible to transform the general equation in which the coefficients of x_i^2 are supposed to be all zero into a form in which the coefficient of x'^2_1 is other than zero.

Secondly, we notice that, without loss of generality, we may assume that *any one* of the four quantities $a_{11}, a_{22}, a_{33}, a_{44}$ is other than zero. For, suppose, for the sake of definiteness, that $a_{11} = 0$; then, by what we have just seen, we may assume that at least one of the quantities a_{22}, a_{33}, a_{44} is other than zero ; let $a_{kk} \neq 0$ when k has a definite value out of 2, 3, 4. Apply the collineation

$$x_1 = x'_k, \quad x_k = x'_1, \quad x_j = x'_j, \quad 1 \neq k \neq j$$

This transforms the general equation (15.9) into a form in which the coefficient of x'^2_1 is other than zero. That is, by a suitable collineation we may transform the general equation in which one of the coefficients of $x_1^2, x_2^2, x_3^2, x_4^2$ is supposed to be zero into a form in which that particular coefficient is other than zero.

We now prove the following theorem :

Theorem. The equation of a quadric can be transformed by collineation into one of the forms :

$$\sum_{i=1}^{4} a_i x_i^2 = 0, \qquad a_1, a_2, a_3, a_4 = +1, 0, -1$$

Proof : To start with, we may suppose, by what we have just seen, that the given general equation

$$F(x_1, x_2, x_3, x_4) \equiv \sum_{i,j=1}^{4} a_{ij} x_i x_j = 0, \quad a_{ij} = a_{ji}$$

is one in which the coefficient $a_{11} \neq 0$. This function F can then be written as

$$a_{11}\left(x_1^2 + \frac{2}{a_{11}} \sum_{j=2}^{4} a_{1j} x_1 x_j\right) + \sum_{j,k=2}^{4} a_{jk} x_j x_k$$

$$= a_{11}\left\{\left(\frac{1}{a_{11}} \sum_{i=1}^{4} a_{1i} x_i\right)^2 - \left(\frac{1}{a_{11}} \sum_{j=2}^{4} a_{1j} x_j\right)^2\right\} + \sum_{j,k=2}^{4} a_{jk} x_j x_k$$

Apply the collineation

$$x'_1 = \frac{1}{a_{11}} \sum_{i=1}^{4} a_{1i} x_i, \qquad x'_j = x_j, \qquad j = 2, 3, 4$$

Then the given polynomial is transformed as

$$F \longrightarrow a_{11} x_1'^2 + F_1(x'_2, x'_3, x'_4),$$

where F_1 is a homogeneous quadratic function in the variables x_2', x_3', x'_4. Let

$$\cdot \quad F_1 \equiv \sum_{i,j=2}^{4} a'_{ij} x'_i x'_j$$

If in the quadratic form F_1 the coefficients a'_{ij} do not all vanish, we may suppose, as we have seen earlier, that the coefficient of $x_2'^2$ is not zero. Then applying the collineation

$$x''_2 = \frac{1}{a'_{22}} \sum_{i=2}^{4} a'_{2i} x'_i, \quad x'_j = x'_j, \quad j = 1, 3, 4,$$

we see, as before,

$$a_{11} x_1'^2 + F_1 \longrightarrow a_{11} x_1''^2 + a'_{22} x_2''^2 + F_2(x''_3, x''_4),$$

where F_2 is a homogeneous quadratic function in the variables x''_3, x''_4. Again, if in the quadratic form F_2 the coefficients do not all vanish, we may repeat the above process and ultimately transform

$$F \longrightarrow a_{11}x_1'''^2 + a'_{22}x_2'''^2 + a''_{33}x_3'''^2 + a''_{44}x_4'''^2$$

We write the reduced form as

$$b_1x_1'''^2 + b_2x_2'''^2 + b_3x_3'''^2 + b_4x_4'''^2,$$

where the b's are all real. Finally, apply the following transformation :

$$x_i^{iv} = \sqrt{|b_i|}\, x_i''', \quad \text{for } b_i \neq 0,$$

$$x_j^{iv} = x_j''', \quad \text{for } b_j = 0$$

The given equation of the quadric now takes one of the required *normal forms* (dropping the dashes) :

$$\sum a_i x_i^2 = 0, \qquad a_i = +1, 0, -1 \tag{15.10}$$

Hence the theorem. It may be noticed that in all the collineations that we have used, the determinants of the coefficients are different from zero.

We now consider the different cases that may arise in (15.10) and obtain the follwing *projective classification of quadrics* :

$$x_1^2 = 0, \tag{15.11}$$

representing *two coincident planes* (or plane fields).

$$x_1^2 + x_2^2 = 0, \tag{15.12}$$

representing *a line* (or a row of points), or a pair of planes without real trace but with a real line of intersection.

$$x_1^2 - x_2^2 = 0, \tag{15.13}$$

representing *a pair of planes*.

$$x_1^2 + x_2^2 + x_3^2 = 0, \tag{15.14}$$

representing *a point* or a cone without real trace but with a real vertex.

$$x_1^2 + x_2^2 - x_3^2 = 0, \tag{15.15}$$

representing *a cone of the second degree*. A *cone* is a surface generated by a line which passes through a fixed point, called the vertex, and through the points of a fixed curve. In order to see that (15.15) is a quadric cone, we take the section of the surface by the plane $x_3 = k$ and obtain a conic

$$x_1^2 + x_2^2 = k^2, \quad x_3 = k$$

Any point on this conic has the coordinates

$$(r, \sqrt{k^2-r^2}, k, \lambda)$$

Therefore the coordinates of points on the line joining this point and the vertex $(0, 0, 0, \mu)$ are given by

$$(r, \sqrt{k^2-r^2}, k, \lambda-\mu\nu)$$

Since all these points satisfy (15.15), the surface is a quadric cone. To resume, we have the further normal forms

$$x_1^2+x_2^2+x_3^2+x_4^2 = 0, \tag{15.16}$$

representing *a quadric without real trace*.

$$x_1^2+x_2^2+x_3^2-x_4^2 = 0, \tag{15.17}$$

representing *a quadric* (without any speciality).

$$x_1^2+x_2^2-x_3^2-x_4^2 = 0, \tag{15.18}$$

representing *a ruled quadric*. A *ruled surface* is a surface generated by the motion of a line in space. In order to see that (15.18) is a ruled quadric, let the equation be written as

$$(x_1+x_3)(x_1-x_3) = (x_4+x_2)(x_4-x_2)$$

This surface is therefore the locus of either of the lines

$$(x_1+x_3) = \mu(x_4+x_2), \quad \mu(x_1-x_3) = (x_1-x_2)$$

and
$$(x_1-x_3) = \nu(x_4+x_2), \quad \nu(x_1+x_3) = (x_4-x_2),$$

where μ, ν are parameters not equal to zero. It is obviously impossible to assign values to μ and ν so that the equations of the two lines become identical. Hence the two systems of lines are distinct and the surface (15.18) can be generated by either system. These lines are called the *generators* or *rulings* of the surface.

The above classification may also be considered in the following manner :

1. $|a_{ij}| = 0$.

(*i*) If the rank of the matrix (a_{ij}) is one, the quadric consists of two coincident planes, as in (15.11).

(*ii*) If the rank of (a_{ij}) is two, the quadric consists of a row of points or a pair of planes, as in (15.12), (15.13).

(*iii*) If the rank of (a_{ij}) is three, the quadric consists of a point or a cone, as in (15.14), (15.15).

2. $| a_{ij} | \neq 0$.

(*i*) If $| a_{ij} | > 0$, the quadric is either without real trace or has real generating lines, as in (15.16), (15.18).

(*ii*) If $| a_{ij} | < 0$, the quadric has real trace but no (real) generating line, as in (15.17).

62. Projective generation of ruled quadrics. Given two triads of points A, B, C and A', B', C' on two lines s and s' which are skew. We have seen in § 55 that we can set up a unique projectivity between the rows $(ABC\ldots)$ and $(A'B'C'\ldots)$ such that (A, A'), (B, B'), (C, C') are pairs of corresponding points. The geometrical construction for establishing this projectivity has already been given there and it need not be repeated here.

The set of all lines AA', BB', CC', . . . which join pairs of corresponding points of two projective rows whose bases are skew is said to form a *regulus*. The individual lines are called the *generators* and these generators occupy a curved surface in space. Since the rows $(ABC\ldots)$, $(A'B'C'\ldots)$ are prejective, the pencil of planes $s'(ABC\ldots)$, $s(A'B'C'\ldots)$ are also projective. The generators AA', BB', CC', . . . are the lines of intersections of the corresponding planes of these two pencils of planes. Thus, any regulus generated by two projective rows can also be generated by two projective pencils of planes, and vice versa. For, let $(\alpha\beta\gamma\ldots)$, $(\alpha'\beta'\gamma'\ldots)$ be two projective pencils of planes whose axes s, s' are skew; then the rows $s'(\alpha\beta\gamma\ldots)$, $s(\alpha'\beta'\gamma'\ldots)$ are projective and generate a regulus whose generators are $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, . . .

Suppose now that the point A'' on AA' is varied in our construction given in § 55. We then obtain other lines $(s'', s''', \ldots)$ each of which meets all the generators AA', BB', CC', ... Every pair of the rows $(ABC\ldots)$, $(A'B'C'\ldots.)$, $(A''B''C''\ldots)$, $(A'''B'''C'''\ldots.)$, . . . are projective. Further, it follows from the same consideration as above, that every pair of the rows $(AA'A''A'''\ldots.)$, $(BB'B''B'''\ldots.)$, $(CC'C''C'''\ldots)$, ... are projective. Hence the lines $s, s', s'', s''', \ldots$ are the generators of another regulus.

The two systems of the generators of the two reguli possess the property that no two generators of the same system can meet, while each generator of one system meets all the generators of the other system. The generators of the two systems therefore lie on the same surface which is obviously a ruled surface. Through each point of this ruled surface pass two generators, one of each system.

Let us now show that the section of the above ruled surface by an arbitrary plane is a conic (which may be nondegenerate or degenerate).

Let p and q be any two generators of the same system (*i.e.*, belonging to the same regulus). The surface may then be regarded as generated by the projective pencils of planes whose axes are p and q. Consider the section of the surface by an arbitrary plane α. First, suppose that α does not contain any generator. If α meets p and q in the points P and Q, then the section of the two projective pencils of planes by α and two projective (and not perspective) pencils of lines whose centres are P and Q. Hence the section is the conic generated by these two projective pencils of lines (see § 41). Secondly, suppose that α contains a generator p. Then α may be regarded as a plane of the pencil of planes whose axis is p and so there exists a plane α' corresponding to α in the projective pencil of planes whose axis is q. These two corresponding planes α, α' intersect in a line p' which must be a generator. Hence the section consists of the two lines p and p' and is therefore a degenerate conic.

On account of the above property, the surface generated by two projective rows whose bases are skew, or by two projective pencils of planes whose axes are skew, is a ruled quadric.

If P is the point of intersection of two generators p, p' of different systems, then it can be seen that the plane passing through p, p' is the tangent plane to the ruled quadric at P.

Projective generation of quadric cones. It was seen in § 41 that the locus of points of intersections of corresponding rays of two projective pencils of lines $(abc\ldots)$ and $(a'b'c'\ldots)$ whose centres are S and S' is a conic locus passing through S and S'. Now project this figure from an external point P in space. The two projective pencils of lines are projected into two projective pencils of planes in the same bundle (P) and the points aa', bb', cc', . . . are projected into lines of intersections of corresponding planes of the two projective pencils of planes. The conic locus is therefore projected into what is called *conical locus of the second order.* We have thus arrived at the following theorem :

Two pencils of planes $(\alpha\beta\gamma\ldots)$ *and* $(\alpha'\beta'\gamma'\ldots)$ *in the same bundle, which are projective but not perspective or coaxial, generate a conical locus of the second order whose generators* $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, . . . *are the intersections of corresponding planes.*

Further, we had the dual theorem in § 41 that the lines joining corresponding points of two projective rows $(ABC\ldots)$, $(A'B'C'\ldots)$ whose bases are s and s' form a conic envelope, the lines s and s' belonging to the envelope. Now project this figure from an external point P in space. The two projective rows are projected into two projective pencils of lines in the same bundle (P) and the lines AA', BB', CC', . . .

are projected into planes passing through the corresponding rays of the two projective pencils of lines. The conic envelope is therefore projected into what is called a *conical envelope of the second class.* We may thus state the following dual theorem :

Two pencils of lines $(pqr\ldots)$, $(p'q'r'\ldots)$ *in the same bundle, which are projective but not perspective or coplanar, generate a conical envelope of the second class, the planes of the envelope being the planes passing through the corresponding rays.*

Now take the generators $s, s', s'', \ldots$ of one system of a ruled quadric generated by two projective rows whose bases are skew. These generators will meet any two generators of the other system in two projective rows $(ABC\ldots)$, $(A'B'C'\ldots)$. If we project these rows from an external point P in space, we obtain two projective pencils of lines in the same bundle (P). The set of planes PAA', PBB', PCC', . . . are the planes passing through the corresponding rays. Hence these planes form a conical envelope of the second class. Since each of these planes, say PAA', meets the ruled quadric in another generator, say s, the plane PAA' is a tangent plane to the ruled quadric at the point of intersection of AA' and s. The conical envelope is therefore a tangent cone to the ruled quadric.

CHAPTER XVI

POLARITY

63. Correlation. A correlation of the space is given by a transformation of the form

$$\rho u_i = \sum_{j=1}^{4} a_{ij} x_j, \qquad i = 1, 2, 3, 4 \tag{16.1}$$

This is a point-to-plane transformation. Consider the matrix (a_{ij}). If the rank of the matrix is four, *i.e.*, if $| a_{ij} | \neq 0$, there is always a plane which is *correlative* (or corresponding or dual) to a point and the correlation (16.1) is then a one-to-one projective correspondence between the points and the planes of the space. If the rank of the matrix is less than four, *i.e.*, if $| a_{ij} | = 0$, we must have x's, not all zero, for which the four equations

$$\sum_{j=1}^{4} a_{ij} x_j = 0, \qquad i = 1, 2, 3, 4$$

are satisfied simultaneously. There are therefore no correlative planes (u_i) corresponding to those points (x_i) which are the solutions of the above four equations.

Let these four equations define four planes. If the rank of the matrix (a_{ii}) is three, the four planes meet in a point. This point has no correlative plane. If the rank is two, the planes meet in a line and so the points of this line have no correlative planes. If the rank is one, the planes coincide and, for points of this coincident plane, there are no correlative planes. If the rank of (a_{ij}) is zero, no point in the projective space has a correlative plane.

As in § 60, we say that r rows of the matrix (a_{ij}), $r \leq 4$, are *linearly dependent* if r constants λ_i, not all zero, exist such that the four relations

$$\sum \lambda_r a_{ri} = 0, \qquad i = 1, 2, 3, 4 \tag{16.2}$$

hold. For example, the first and the third rows are linearly dependent if

$$\lambda_1 a_{11} + \lambda_3 a_{31} = 0, \quad \lambda_1 a_{12} + \lambda_3 a_{32} = 0,$$
$$\lambda_1 a_{13} + \lambda_3 a_{33} = 0, \quad \lambda_1 a_{14} + \lambda_3 a_{34} = 0$$

If then two rows are dependent, all the rows are dependent and $| a_{ij} | = 0$.

Conversely, if $| a_{ij} | = 0$, the rows are dependent. Similarly r columns of the matrix are linearly dependent if constants λ_r, not all zero, exist such that

$$\sum \lambda_r a_{ir} = 0, \quad i = 1, 2, 3, 4 \qquad (16.2')$$

And if two and therefore all the columns are dependent, the rank of the matrix (a_{ij}) is less than four, and conversely.

Let the rows of the matrix be linearly dependent, so that (16.2) hold. Multiply the four relations (16.2) by x_1, x_2, x_3, x_4 respectively and add. So

$$0 = \sum a_{ik} \lambda_i x_k = \sum u_i \lambda_i, \qquad (16.3)$$

where (u_i) is the plane which is correlative to the point (x_i). The equation (16.3) shows that the point (λ_i) lies on the plane (u_i). Thus the correlative planes of all points, whose correlative planes exist, pass through the point (λ_i).

Conversely, let a point (λ_i) be incident on all correlative planes (u_i) ; then

$$\sum u_i \lambda_i = 0 ;$$

so, from (16.1),

$$\sum a_{ik} x_k \lambda_i = 0$$

Since this must be an identity for all (x_i) whose correlative planes exist, the coefficients of x_i must be separately zero. Therefore the four relations (16.2) must hold. This shows that the rows of (a_{ij}) are linearly dependent.

Again, if two points (λ_i), (μ_i) have the same correlative plane,

$$\sum a_{ik} \lambda_k = \Sigma a_{ik} \mu_k, \quad \text{or} \quad \Sigma a_{ik}(\lambda_k - \mu_k) = 0$$

The columns of (a_{ij}) are therefore linearly dependent, and the point $(\lambda_i - \mu_i)$ has no correlative plane. Conversely, let the columns of (a_{ij}) be linearly dependent. Then there exist λ_i such that (16.2′) holds. This shows that the point (λ_i) has no correlative plane. Adding the equations (16.2) to (16.1),

$$\rho u_i = \Sigma a_{ik} (x_k + \lambda_k)$$

This shows that the points $(x_i + \lambda_i)$ and (x_i) have the same correlative plane. In other words, there are more than one point having the same correlative plane.

64. Polarity and null-system. Let a correlation be defined by (16.1), namely,

$$\rho u_i = \sum_k a_{ik} x_k \qquad i = 1, 2, 3, 4$$

If (ξ_i) is a point incident with the plane (u_i), we must have

$$0 = \Sigma u_i \xi_i = \Sigma a_{ik} x_k \xi_i = \Sigma u'_k x_k, \text{ say,}$$

where

$$\sigma u'_k = \sum_i a_{ik} \xi_i \qquad (16.4)$$

The equations (16.4) define a correlation which is, in general, different from (16.1). The two correlations are such that if (ξ_i) is incident with (u_i), then (x_i) is incident with (u'_i).

If the two correlations (16.1) and (16.4) are identical, then

$$\rho\, a_{ik} = \sigma a_{ki}$$

Whence
$$\rho^2 a_{ik} = \rho\,\sigma a_{ki} = \sigma(\rho\, a_{ki}) = \sigma^2 a_{ik}\ ;$$

therefore
$$\rho^2 = \sigma^2, \text{ or } \rho = \pm\sigma$$

Hence
$$a_{ik} = \pm a_{ki}$$

Taking the upper sign, $a_{ik} = a_{ki}$, and so the matrix (a_{ik}) is symmetrical. In this case the correlation is called a *polarity* ; a point and its correlative plane are called *pole* and *polar*. Taking the lower sign, $a_{ik} = -a_{ki}$, so $a_{ii} = 0$, and the matrix (a_{ik}) is skew-symmetrical. In this case the correlation is called a *null-system*.

By the correlation (16.1), the locus of all points which are incident with their corresponding correlative planes is given by

$$\sum_{i,k} a_{ik}\, x_i\, x_k = 0 \tag{16.5}$$

If the correlation is a polarity, the equation (16.5) represents a quadric surface and this surface is called the *nucleus* of the polarity. If the correlation is a null-system, the coefficients of (16.5) are all zero ; so there is no quadric nucleus of a null-system.

On the other hand, given an equation of the second degree $\Sigma\, a_{ik}\, x_i x_k = 0$ in which the coefficients are not all zero, we may, without loss of generality, put

$$a_{ik} = (a_{ik} + a_{ki})/2, \quad \text{so that} \quad a_{ik} = a_{ki}$$

Then the polarity generated by the given quadric is defined as

$$\rho u_i = \sum_k a_{ik}\, x_k, \quad i = 1, 2, 3, 4, \quad a_{ik} = a_{ki} \tag{16.6}$$

The matrix (a_{ik}) is said to be the matrix of the polarity.

As an *application*, consider the quadric $x_1^2 - d^2 x_4^2 = 0$ (a pair of planes). The matrix of the polarity generated by the quadric is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d^2 \end{pmatrix};$$

This is of rank two. So the points of the line $x_1 = 0 = x_4$ have no polar planes. Further, consider the plane $x_1 + kx_4 = 0$. The coordinates of points of this plane may be taken as $(1, \rho, \sigma, -1/k)$. It follows from the above matrix that the polar planes of these points have the coordinates $(1, 0, 0\ d^2/k)$, independent of ρ and σ. So, these planes are identical and have the equation $x_1 + d^2x_4/k = 0$ and this polar plane passes through the line $x_1 = 0 = x_4$. In particular, each of the two planes

$$x_1 + dx_4 = 0, \quad x_1 - dx_4 = 0,$$

for $k = \pm d$, which constitute the given quadric, is the polar plane of points incident with it.

Let a polarity be given by (16.6). The polarity is *nondegenerate* or *degenerate* according as the determinant $| a_{ik} |$ does not or does vanish. Suppose that the polarity is nondegenerate and let, as usual, A_{ik} be the cofactors of a_{ik} in $| a_{ik} |$. Then, from (16.6), we have the dual nondegenerate polarity

$$\sigma x_i = \sum_k A_{ik} u_k, \quad i = 1, 2, 3, 4, \quad A_{ik} = A_{ki} \tag{16.7}$$

The nucleus of the polarity (16.6) is the locus of points incident with their polars and is given by $\Sigma a_{ik} x_i x_k = 0$; it is a *quadric locus* (a locus of the second order). The nucleus of the polarity 16.7) is the envelope of planes which are incident with their poles and is given by $\Sigma A_{ik}\, u_i\, u_k = 0$; it is a *quadric envelope* (an envelope of the second class). The equation of the envelope is identically the same as

$$\begin{vmatrix} 0 & u_1 & u_2 & u_3 & u_4 \\ u_1 & a_{11} & a_{12} & a_{13} & a_{14} \\ u_2 & a_{21} & a_{22} & a_{23} & a_{24} \\ u_3 & a_{31} & a_{32} & a_{33} & a_{34} \\ u_4 & a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0$$

This equation is also spoken of as the equation in plane coordinates, or the *tangential equation*, of the quadric locus (in point coordinates).

Lastly, as in § 61, the *normal forms* of equations of quadric envelopes are

$$\Sigma\, a_i u_i^2 = 0, \qquad a_i = 1, 0, -1. \tag{16.8}$$

65. Transformation of correlation by collineation. Let a correlation and a collineation be given respectively by

$$\rho u_i = \sum_k a_{ik}\, x_k \tag{16.9}$$

and
$$\sigma x_i = \sum_k b_{ik}\, x_k', \qquad |\, b_{ik}\,| \neq 0 \tag{16.10}$$

The transformation of planes for this collineation is

$$\varpi u_i = \sum_k B_{ik}\, u'_k,$$

where B_{ik} are the cofactors of b_{ik} in $|\, b_{ik}\,|$. Then

$$\rho' \sum_k B_{ik}\, u'_k = \rho u_i = \sum_k a_{ik}\, x_k = \sigma' \sum_{k,j} a_{ik}\, b_{kj}\, x'_j,$$

or
$$\varpi' \sum_k B_{ik}\, u'_k = \sum_{k,j} a_{ik}\, b_{kj}\, x'_j$$

Multiply both sides by b_{ir} and sum for i. So

$$\varpi' \sum_{i,k} b_{ir}\, B_{ik}\, u'_k = \sum_{i,k,j} b_{ir}\, a_{ik}\, b_{kj}\, x'_j$$

Now we have
$$\sum_{i,k} b_{ir}\, B_{ik}\, u'_k = |\, b_{ik}\,|\, u'_r$$

Put
$$c_{rj} = \sum_{i,k} b_{ir}\, b_{kj}\, a_{ik} \tag{16.11}$$

Therefore we have ultimately

$$\rho u'_r = \sum_j c_{rj}\, x'_j \tag{16.12}$$

Accordingly the transformation (16.12) is the correlation in the transformed coordinates when (16.10) is applied to (16.9) and c_{rj} is given by (16.11).

Interchanging the suffixes r and j in (16.11) and then interchanging i and k (which is permissible under the summation), we have

$$c_{jr} = \sum_{i,k} b_{ij}\, b_{kr}\, a_{ik} = \sum_{i,k} b_{kj}\, b_{ir}\, a_{ki}$$

Hence
$$c_{rj} = c_{jr}, \quad \text{if} \quad a_{ik} = a_{ki}$$

and
$$c_{rj} = -c_{jr}, \quad \text{if} \quad a_{ik} = -a_{ki}$$

Thus *a polarity and a null-system are transformed respectively into a polarity and a null-system by a collineation.*

Further, it follows from (16.11) that

$$\sum_s B_{ms}\, c_{rs} = \sum_{s,i,k} B_{ms}\, b_{ir}\, b_{ks}\, a_{ik} = |\, b_{ik}\,| \sum_i b_{ir}\, a_{im}$$

Let y_m, $m = 1, 2, 3, 4$, be four quantities not all zero. Multiplying both sides by y_m and summing for m we have

$$\sum_{s,m} B_{ms}\, c_{rs}\, y_m = |\, b_{ik}\,| \sum_{i,m} b_{ir}\, a_{im}\, y_m$$

Put
$$\sum_m B_{ms}\, y_m = z_s, \quad s = 1, 2, 3, 4 \tag{16.13}$$

So finally
$$\sum_s c_{rs}\, z_s = |\, b_{ik}\,| \sum_{i,m} b_{ir}\, a_{im}\, y_m \tag{16.14}$$

Now if (y_1, y_2, y_3, y_4) is a solution of the system of four equations

$$\sum_m a_{im} y_m = 0, \quad i = 1, 2, 3, 4,$$

then, from (16.14), (z_1, z_2, z_3, z_4) is a solution of the system of four equations

$$\sum_s c_{rs} z_s = 0, \quad r = 1, 2, 3, 4$$

But it follows from (16.13) that the system of solutions (z) and the system of solutions (y) are of the same rank. Hence the rank of the matrix (c_{ij}) is the same as the rank of the matrix (a_{ij}).

Again, from (16.11),

$$| c_{rj} | = | b_{ir} | \; | b_{kj} | \; | a_{ik} | = | b_{ij} |^2 \; | a_{ik} |$$

Therefore, $| c_{rj} |$ has the same sign as $| a_{ik} |$. And these two determinants are the determinants of the coefficients of the original and the transformed correlations (16.9), (16.12).

Thus *the rank of the matrix and the sign of the determinant of the coefficients of a correlation are not altered by a collineation.*

As an *application*, consider the quadric

$$x_1^2/a^2 - x_2^2/b^2 - 2px_3x_4 = 0, \quad abp \neq 0$$

The matrix of the polarity generated by the quadric is, by what we have seen in the last article,

$$\begin{pmatrix} 1/a^2 & 0 & 0 & 0 \\ 0 & -1/b^2 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & -p & 0 \end{pmatrix}$$

This matrix is of rank four and its determinant has the value $p^2/a^2b^2 > 0$. When the quadric is transformed by collineation, the rank of the matrix as well as the sign of the determinant will remain unchanged. Let now the quadric be transformed into the normal form. Its equation will take one of the forms (§ 61)

$$\sum_i a_i x_i^2 = 0, \quad a_i = 1, 0, -1$$

Since the rank of the matrix and the sign of the determinant have to remain unaltered, the matrix has to be of the type

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

But since the quadric is one with real trace, the matrix has to be the second one. Hence the surface is a ruled surface.

66. Polar fields. A polarity is said to establish a *polar field*. Consider a polarity in space

$$\rho u_i = \sum_k a_{ik} x_k, \quad i = 1, 2, 3, 4, \quad a_{ik} = a_{ki} \tag{16.15}$$

If a point (x'_i) lies on the polar plane of (x_i), then

$$\sum_{i,k} a_{ik} x_i x'_k = 0 \tag{16.16}$$

This shows that (x_i) lies on the polar plane of (x'_i). The two points (x_i) and (x'_i), as well as their polars, are conjugate. On the other hand, suppose that we are given the relation (16.16) in which $a_{ki} = a_{ik}$. Then we may say that there exists a polar field given by (16.15) in which (x_i) and (x'_i) are conjugate points.

Let us start with (16.16). Take a plane α defined by the three points (ξ_{1i}), (ξ_{2i}), (ξ_{3i}). The parametric equations of α are then, by (13.7),

$$x_i = \sum_{\mu=1}^{3} y_\mu \xi_{\mu i}, \qquad i = 1, 2, 3, 4,$$

where y_1, y_2, y_3 are three parameters. For each set of values of (y_1, y_2, y_3), other than all zero, we obtain a point on α and proportional values of y represent the same point. Also, for each of $y_1 = 0$, $y_2 = 0$, $y_3 = 0$, we obtain a line in α. Therefore (y_1, y_2, y_3) may be considered as the homogeneous coordinates of a point in α. Let (y'_1, y'_2, y'_3) refer to (x'_1). (We are here considering those of the points (x_i) and (x'_i) satisfying (16.16) as lie on α). Then we have from (16.16)

$$0 = \sum_{i,k=1}^{4} a_{ik} x_i x'_k = \sum_{\mu,\nu=1}^{3} \sum_{i,k=1}^{4} a_{ik} y_\mu \xi_{\mu i} y'_\nu \xi_{\nu k}$$

Put

$$b_{\mu\nu} = \sum_{i,k=1}^{4} a_{ik} \xi_{\mu i} \xi_{\nu k} \tag{16.17}$$

Hence finally

$$\sum_{\mu,\nu=1}^{3} b_{\mu\nu} y_\mu y'_\nu = 0 \tag{16.18}$$

Interchanging the suffixes μ and ν in (16.17), and then interchanging i and k (which is permissible under the summation) and remembering that $a_{ik} = a_{ki}$, we have

$$b_{\nu\mu} = \sum_{i,k} a_{ik} \xi_{\nu i} \xi_{\mu k} = \sum_{i,k} a_{ik} \xi_{\nu k} \xi_{\mu i}$$

Therefore

$$b_{\mu\nu} = b_{\nu\mu}$$

Thus the equation (16.18), which holds for points in the plane α, is exactly of the same nature as the equation (16.16) which holds for points in the space. Therefore (see § 37) there exists a polar field in α in which (y_i) and (y'_i) are conjugate points and this polar field in the plane is generated by the polar field in the space.

Hence, *the section of a polar field in space by a plane is a polar field in the plane* ; and this polar field in the plane exists whether the polar field in the space is nondegenerate or degenerate. Geometrically, this means that the section of the nucleus of the polarity in the space by α is the nucleus of the polarity in α.

Again, let us take a line g as the join of two points (η_{1i}), (η_{2i}). The parametric equations of the line are, by (13.6),

$$x_i = \sum_{\mu=1}^{2} z_\mu \, \eta_{\mu i}, \qquad i = 1, 2, 3, 4,$$

where z_1, z_2 are two parameters. For each set of values of (z_1, z_2), other than both zero, we obtain a point of g and proportional values of z represent the same point. Therefore (z_1, z_2) can be considered as the homogeneous coordinates of a point of g. Let (z'_1, z'_2) refer to (x'_i). (We are considering here those of the points (x_i) and (x'_i) satisfying (16.16) as lie on g). Then we have from (16.16)

$$0 = \sum_{i,k=1}^{4} a_{ik} \, x_i \, x'_k = \sum_{\mu,\nu=1}^{2} \sum_{i,k=1}^{4} a_{ik} \, z_\mu \, \eta_{\mu i} \, z'_\nu \, \eta_{\nu k}$$

Put

$$c_{\mu\nu} = \sum_{i,k=1}^{4} a_{ik} \, \eta_{\mu i} \, \eta_{\nu k}$$

Then finally

$$\sum_{\mu,\nu=1}^{2} c_{\mu\nu} \, z_\mu \, z'_\nu = 0, \tag{16.19}$$

where, as before,

$$c_{\mu\nu} = c_{\nu\mu}$$

Thus the equation (16.19), which holds for points of the line g, is exactly of the same nature as the equation (16.18) which holds for points of the plane α and the equation (16.16) which holds for points of the space. Therefore we can say that there exists a polar field in g in which (z_i) and (z'_i) are conjugate points and this polar field in the line is generated by the polar field in the space. Geometrically this means that the conjugate of a given point of g is the point of intersection of g with the polar plane of the given point with respect to the polarity in space.

Hence, *the section of a polar field in space by a line is a polar field in the line.*

The polarity (16.19) in g can be written as

$$(c_{11}z'_1 + c_{12}z'_2)z_1 + (c_{12}z'_1 + c_{22}z'_2)z_2 = 0$$

So we may write

$$\rho z_1 = c_{12}z'_1 + c_{22}z'_2$$
$$\rho z_2 = -c_{11}z'_1 - c_{12}z'_2$$

Therefore if $c_{11}c_{22} - c_{12}^2 \neq 0$, the polarity is an *involution* of points of g. Two corresponding points are therefore conjugate points, and the points in this polarity (or involution) which are self-conjugate (or double points) are given by

$$c_{11}z_1^2 + 2c_{12}z_1z_2 + c_{22}z_2^2 = 0$$

If there are two real and distinct solutions, the polarity is a *hyperbolic* involution. If there is no real solution the polarity is an *elliptic* involution.

If $c_{11}c_{22} - c_{12}^2 = 0$, there are two identical solutions of the above equation and the polarity is then degenerate. In the case, let the rank of (c_{ik}) be one. We may, without loss of generality, suppose that $c_{11} \neq 0$. So we may put

$$c_{11} = 1, \quad c_{12} = a, \quad c_{22} = a^2, \quad a \neq 0$$

Then the polarity (16.19) becomes

$$(z_1 + az_2)(z'_1 + az'_2) = 0\,;$$

therefore either $z_1 + az_2 = 0$ or $z'_1 + az'_2 = 0$

Hence, one of the conjugate points is fixed and the other arbitrary. Geometrically this means that the given polarity in space is such that the polar planes of all points of the line g pass through a fixed point of g.

Thus there may arise four cases of the polarity in a line depending on the position of the line in the polar field in the space

$$\left.\begin{array}{ll}(i) & \text{hyperbolic involution}\\ (ii) & \text{elliptic involution}\end{array}\right\}; \quad \text{rank } (c_{ik}) = 2$$

(*iii*) all points have a fixed conjugate point ; rank $(c_{ik}) = 1$

(*iv*) no polarity at all ; rank $(c_{ik}) = 0$.

CHAPTER XVII

GEOMETRY IN THE EXTENDED CARTESIAN SPACE

67. The circle at infinity. Let Γ be a polarity in space and α, β two planes which intersect in a line g. We have seen in the last article that Γ generates a polarity Γ_1 in α and a polarity Γ_2 in β and each of the polarities Γ, Γ_1, Γ_2 generate the same polarity in g.

On the other hand, suppose we have in a given plane, say $x_4 = 0$, a given polarity whose matrix is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \qquad a_{ij} = a_{ji}$$

Then all polarities in space whose matrices are

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a \\ a_{21} & a_{22} & a_{23} & b \\ a_{31} & a_{32} & a_{33} & c \\ a & b & c & d \end{pmatrix},$$

where a, b, c, d are arbitrary quantities, generate the given polarity in the given plane $x_4 = 0$.

Further, suppose that we have in another given plane, say $x_3 = 0$, another given polarity whose matrix is

$$\begin{pmatrix} b_{11} & b_{12} & b_{14} \\ b_{21} & b_{22} & b_{24} \\ b_{41} & b_{42} & b_{44} \end{pmatrix}, \qquad b_{ij} = b_{ji}$$

The two given polarities in the plane $x_4 = 0$ and $x_3 = 0$ will, in general, generate different polarities on the line of intersection of the two planes. But supposing that the two polarities on the common line is the same, we may, without loss of generality, put

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Then all those polarities in space whose matrices are

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_{14} \\ a_{21} & a_{22} & a_{23} & b_{24} \\ a_{31} & a_{32} & a_{33} & c \\ b_{41} & b_{42} & c & b_{44} \end{pmatrix},$$

where c is an arbitrary quantity, generate the given polarities in the given planes and the same polarity on their common line.

In projective geometry all planes are equivalent, so are all lines and all points. Suppose now we digress from this generality and *specialise one plane* and call this plane the plane at infinity. All lines and points of this plane shall be called the lines and points at infinity. Hence the intersections of other planes and lines with the plane at infinity are lines and points at infinity. Let the equation of the plane at infinity be $x_4 = 0$. It is to be noted that when the specialisation has thus been made, we may pass from the homogeneous coordinates (x_1, x_2, x_3, x_4) to the nonhomogeneous coordinates (x, y, z) by setting

$$x = x_1/x_4, \quad y = x_2/x_4, \quad z = x_3/x_4$$

for all points which are not points at infinity.

Further, suppose we take a *special polarity* in the plane at infinity defined by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The nucleus of this polarity is the second degree curve

$$x_1^2 + x_2^2 + x_3^2 = 0 \tag{17.1}$$

in the plane $x_4 = 0$. The equation (17.1) represents a curve without real trace and this curve is called *the circle at infinity*. All polarities in space whose matrices are

$$\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ a & b & c & d \end{pmatrix},$$

where a, b, c, d are arbitrary quantities, generate the special polarity in the plane at infinity. The nuclei of all these polarities in space are the quadrics

$$x_1^2 + x_2^2 + x_3^2 + dx_4^2 + 2ax_1x_4 + 2bx_2x_4 + 2cx_3x_4 = 0 \tag{17.2}$$

The section of any of these quadrics by the plane at infinity is the circle at infinity. In nonhomogeneous coordinates, the quadrics take the form

$$(x+a)^2 + (y+b)^2 + (z+c)^2 - r^2 = 0,$$

where

$$r^2 = a^2 + b^2 + c^2 - d$$

The quadrics (17.2) are called *spheres*. A sphere is without real trace if $r^2 < 0$ and is a point-sphere if $r = 0$.

Thus *the section of every sphere by the plane at infinity is the circle at infinity and the spheres are those quadrics which generate the special polarity in the plane at infinity*. [Cf : the section of every circle in a plane by the line at infinity of that plane consists of the two circular points at infinity and the circles are those conics which generate the same (special) involution in the line at infinity, in § **44**.]

Again, take a plane, say $x_3 = 0$. The section of any one of the spheres (17.2) by this plane is a circle

$$x_1^2 + x_2^2 + dx_4^2 + 2ax_1x_4 + 2bx_2x_4 = 0$$

The polarity generated by this circle in the plane $x_3 = 0$ is defined by the matrix

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ a & b & d \end{pmatrix}$$

And therefore the polarity generated by the above polarity in $x_3 = 0$ in the line $x_3 = 0 = x_4$ is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But this is the same polarity as generated by the special polarity in the plane at infinity on the same line.

Hence *a circle has the property that it generates in the line at infinity of its plane the same polarity as does the circle at infinity in the same line.*

68. Orthogonality as a polarity. Consider the special polarity in the plane at infinity $x_4 = 0$, namely

$$\rho u_1 = x_1, \quad \rho u_2 = x_2, \quad \rho u_3 = x_3\,;$$

or

$$\sigma x_1 = u_1, \quad \sigma x_2 = u_2, \quad \sigma x_3 = u_3,$$

where (x_1, x_2, x_3) and (u_1, u_2, u_3) are the point and line coordinates. The nucleus of this polarity of the second order is $x_1^2+x_2^2+x_3^2 = 0$ and of the second class is $u_1^2+u_2^2+u_3^2 = 0$.

With respect to this polarity, two conjugate points (x_i) and (y_i) and two conjugate lines (u_i) and (v_i) satisfy the relations

$$x_1y_1+x_2y_2+x_3y_3 = 0$$

and

$$u_1v_1+u_2v_2+u_3v_3 = 0 \tag{17.3}$$

On the other hand, suppose we are given the relation (17.3) between the first three coordinates of two planes whose equations in point coordinates are

$$u_1x_1+u_2x_2+u_3x_3+u_4x_4 = 0$$

and

$$v_1x_1+v_2x_2+v_3x_3+v_4x_4 = 0$$

In nonhomogeneous coordinates the last two equations are

$$u_1x+u_2y+u_3z+u_4 = 0$$

and

$$v_1x+v_2y+v_3z+v_4 = 0$$

So, by virtue of the relation (17.3), the planes are orthogonal. But (u_1, u_2, u_3) and (v_1, v_2, v_3) may be regarded as the line coordinates in the plane at infinity of the two lines of intersection of the planes (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4) respectively with the plane at infinity.

Therefore, *if the lines in which two distinct planes intersect the plane at infinity are conjugate to one another with respect to the circle at infinity, then the two planes are orthogonal to one another.* (Cf : two straight lines are orthogonal when they are harmonically separated by the isotropic lines passing through their common point, in § 14.1.)

Consider then four planes $\alpha, \beta, \gamma, \delta$ of a pencil whose coordinates are

$$(u_i),\ (v_i),\ (\mu u_i+\nu v_i),\ (\mu' u_i+\nu' v_i)$$

If the first and the third planes, as also the second and the fourth planes, are orthogonal, we have, as above, for $i = 1, 2, 3$,

$$\mu \Sigma u_i^2+\nu \Sigma u_i v_i = 0, \quad \mu' \Sigma u_i v_i+\nu' \Sigma v_i^2 = 0$$

Therefore the cross-ratio

$$(\alpha\beta, \delta\gamma) = \mu\nu'/\nu\mu'$$

$$= \frac{(\Sigma u_i v_i)^2}{(\Sigma u_i^2)(\Sigma v_i^2)} = \cos^2(\alpha, \beta) \tag{17.4}$$

The cross-ratio of the four planes so chosen is therefore equal to the square of the cosine of the angle between the planes α, β.

69. Affine properties of quadrics. If the collineation

$$\rho x'_i = \Sigma\, c_{ij} x_j, \quad i = 1, 2, 3, 4, \quad | c_{ij} | \neq 0$$

leaves the plane at infinity $x_4 = 0$ fixed, we must have

$$c_{41} = c_{42} = c_{43} = 0$$

Therefore the collineation is an affinity [see (14.8) for $n = 3$]. If, moreover, the circle at infinity $x_1^2 + x_2^2 + x_3^2 = 0$, $x_4 = 0$ is to remain fixed, the special polarity of § 67 has also to remain fixed. So, if (x_i), (y_i) are two conjugate points in the plane at infinity and (x'_i), (y'_i) are their transforms by the above affinity, then

$$(x_1 y_1 + x_2 y_2 + x_3 y_3) \longrightarrow \sigma(x'_1 y'_1 + x'_2 y'_2 + x'_3 y'_3)$$

But, for $i, j, k = 1, 2, 3$,

$$\varpi \sum_i x'_i y'_i = \sum_{i,j,k} c_{ij}\, c_{ik}\, x_j\, y_k$$

Therefore
$$\sum_{i,j,k=1}^{3} c_{ij}\, c_{ik} = \left[\begin{array}{ll} \text{a non-zero constant,} & \text{if } j = k \\ 0, & \text{if } j \neq k \end{array} \right.$$

Hence the coefficients c_{ij}, $i, j = 1, 2, 3$, are those of similarity transformation [see (14.9') for $n = 3$].

Thus *the circle at infinity remains fixed under similarity transformations.*

We have seen at the end of § 59 that the equation of a quadric, for which the tetrahedron of reference with vertices are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) is a polar tetrahedron, is

$$a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + a_{44} x_4^2 = 0 \tag{17.5}$$

Now the *centre* of a central quadric is defined as the point in which every chord of the surface which passes through it is bisected. If the quadric (17.5) has a centre and the coordinates of the centre are chosen as (0, 0, 0, 1), the equation of a central quadric can be written as

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + x_4^2 = 0 \tag{17.6}$$

The polarity generated by this quadric is

$$\rho u_1 = a_1 x_1, \quad \rho u_2 = a_2 x_2, \quad \rho u_3 = a_3 x_3, \quad \rho u_4 = x_4$$

The polar of the centre is therefore the plane (0, 0, 0, 1), *i.e.*, the plane at infinity, and the three vertices of the fundamental tetrahedron, other than the centre, lie on this plane. The section of the quadric (17.6) by the plane at infinity is the conic

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 0, \quad x_4 = 0 \quad .$$

It is a *conic at infinity.* In the plane at infinity, this conic generates the polarity

$$\rho u_1 = a_1 x_1, \quad \rho u_2 = a_2 x_2, \quad \rho u_3 = a_3 x_3$$

The triangle of reference, *i.e.*, the triangle with the vertices (1, 0, 0), (0, 1, 0), (0, 0, 1), is a polar triangle with respect to this polarity. On the other hand, the circle at infinity generates the polarity

$$\sigma u_1 = x_1, \quad \sigma u_2 = x_2, \quad \sigma u_3 = x_3$$

The triangle of reference is therefore a polar triangle with respect to this latter polarity.

Thus *a conic at infinity and the circle at infinity have a common polar triangle.*

If the section of a central quadric by the plane at infinity is a real conic, the quadric is called a *hyperboloid* and if the section is a conic without real trace, the quadric is called an *ellipsoid.* If the plane at infinity is incident with its pole with respect to a quadric, the quadric is called a *paraboloid.* (Cf : a conic is a hyperbola, a parabola or an ellipse according as it is met by the line at infinity in two distinct points, in two coincident points or in no point, in § 43.)

CHAPTER XVIII

ORTHOGONAL TRANSFORMATION AND AFFINITY

70. Change of coordinate axes. We now pass on from the system of extended Cartesian space of the last chapter to the ordinary Euclidean space. This is done by withdrawing from the system the plane at infinity. For the Euclidean space, we use non-homogeneous coordinates in which, for simplicity's sake, a point is defined by three coordinates (x, y, z) referred to a right-handed system of three mutually orthogonal axes of coordinates, as in § 45.

Let the equations of three mutually orthogonal planes be given in Hessian normal forms and let these three equations be denoted by $x' = 0$, $y' = 0$, $z' = 0$. Then we have

$$\begin{aligned} x' &= a_1x + b_1y + c_1z + d_1 \\ y' &= a_2x + b_2y + c_2z + d_2 \\ z' &= a_3x + b_3y + c_3z + d_3 \end{aligned} \tag{18.1}$$

where

$$a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2 = a_3^2 + b_3^2 + c_3^2 = 1, \tag{18.1'}$$

$$a_1a_2 + b_1b_2 + c_1c_2 = a_1a_3 + b_1b_3 + c_1c_3 = a_2a_3 + b_2b_3 + c_2c_3 = 0$$

The transformation (18.1) with (18.1′) is known as an *orthogonal transformation* of the space (§ 57). The lines of intersections of the planes $x' = 0$, $y' = 0$, $z' = 0$ form a new system of coordinate axes, the positive x'-, y'-, z'- axes being in the directions of the unit vectors (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) respectively. Let α, β, γ be the unit vectors in the directions of positive x-, y-, z- axes respectively, and similarly let α', β', γ' be the unit vectors in the directions of positive x'-, y'-, z'- axes. Then the nine quantities a_i, b_i, c_i are the following scalar products (§ 46) :

$$\begin{aligned} a_1 &= \alpha.\alpha', & b_1 &= \beta.\alpha', & c_1 &= \gamma.\alpha' \\ a_2 &= \alpha.\beta', & b_2 &= \beta.\beta', & c_2 &= \gamma.\beta' \\ a_3 &= \alpha.\gamma', & b_3 &= \beta.\gamma', & c_3 &= \gamma.\gamma' \end{aligned}$$

Let α'', β'', γ'' be the following vector products (§ 48.1) :

$$\beta' \times \gamma' = \alpha'', \quad \gamma' \times \alpha' = \beta'', \quad \alpha' \times \beta' = \gamma''$$

Then

$$\alpha' = \pm\alpha'', \quad \beta' = \pm\beta'', \quad \gamma' = \pm\gamma''$$

So, the new system of axes is right-handed or left-handed according as we choose the upper or the lower sign.

Let

$$D \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and A_i, B_i, C_i be the cofactors of a_i, b_i, c_i respectively in D. Then the coordinates of the vectors $\alpha'', \beta'', \gamma''$ are (A_1, B_1, C_1), (A_2, B_2, C_2), (A_3, B_3, C_3) respectively. Also, by virtue of the condition (18.1′),

$$D^2 = 1, \quad \text{and} \quad A_i = a_i D, \quad B_i = b_i D, \quad C_i = c_i D$$

(*i*) Taking $D = +1$,

$$a_i = A_i, \quad b_i = B_i, \quad c_i = C_i$$

and the system α', β', γ', and therefore the new system of axes, form a right-handed system.

(*ii*) Taking $D = -1$,

$$a_i = -A_i, \quad b_i = -B_i, \quad c_i = -C_i$$

and the new system of axes form a left-handed system. Finally, we can solve the equations (18.1) and express x, y, z linearly in terms of x', y', z'; the resulting equations are given by

$$\begin{aligned} x &= a_1 x' + a_2 y' + a_3 z' - \Sigma d_i a_i \\ y &= b_1 x' + b_2 y' + b_3 z' - \Sigma d_i b_i \\ z &= c_1 x' + c_2 y' + c_3 z' - \Sigma d_i c_i \end{aligned} \qquad (18.2)$$

Since a_1, a_2, a_3 are the cosines of the angles between the vector α and the vectors α', β', γ' (and similarly for the quantities b_i, c_i) and also since the old axes form an orthogonal system,

$$\begin{aligned} &a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2 = c_1^2 + c_2^2 + c_3^2 = 1, \\ &a_1 b_1 + a_2 b_2 + a_3 b_3 = a_1 c_1 + a_2 c_2 + a_3 c_3 = b_1 c_1 + b_2 c_2 + b_3 c_3 = 0 \end{aligned} \qquad (18.2')$$

We have thus arrived at the following conclusion :

The transformation from one orthogonal coordinate system to another is given by (18.1) with the relations (18.1′). As a consequence, there exists the inverse transformation (18.2) with the relations (18.2′). The square of the determinant D of the coefficients is unity. If D = +1, the sense of both the coordinate systems is the same (e.g., both right-handed) and the transformation is called a rigid motion ; if D = −1, the senses of the two systems of coordinates are opposite and the transformation is called a symmetry.

71. Rigid motion and symmetry. Let us consider the fixed points of the transformation (18.1) with (18.1′). The fixed points of the transformation are the solutions of the equations

$$\begin{aligned}(a_1-1)x+b_1y+c_1z+d_1&=0\\ a_2x+(b_2-1)y+c_2z+d_2&=0\\ a_3x+b_3y+(c_3-1)z+d_3&=0\end{aligned}$$

Let

$$\Delta \equiv \begin{vmatrix} a_1-1 & b_1 & c_1 \\ a_2 & b_2-1 & c_2 \\ a_3 & b_3 & c_3-1 \end{vmatrix}$$

There exists then a solution of the above equations if $\Delta \neq 0$, and there exists either no solution or an infinite number of solutions if $\Delta = 0$. Developing the determinant,

$$\Delta = D-(A_1+B_2+C_3)+(a_1+b_2+c_3)-1 \tag{18.3}$$

1. *Case of rigid motion,* $D = +1$

Here $$a_1 = A_1,\quad b_2 = B_2,\quad c_3 = C_3$$

So, from (18.3), $$\Delta = 0$$

Hence, *in a rigid motion there exists either no fixed point or an infinity of fixed points.*

In particular, if $a_1 = b_2 = c_3 = 1$, the remaining a's, b's, c's vanish and the transformation (18.1) takes the form

$$\begin{aligned}x' &= x+d_1\\ y' &= y+d_2\\ z' &= z+d_3\end{aligned} \tag{18.4}$$

If d_1, d_2, d_3 are not all zero, the transformation (18.4) is called a *parallel displacement* or a *translation.* If $d_1 = d_2 = d_3 = 0$, the transformation (18.4) is called the *identity.* It is obvious that there cannot be any fixed point under parallel displacement whereas all points remain fixed by the identity.

Consider a rigid motion in which the origin remains fixed :

$$\begin{aligned}x' &= a_1x+b_1y+c_1z\\ y' &= a_2x+b_2y+c_2z\\ z' &= a_3x+b_3y+c_3z\end{aligned} \tag{18.5}$$

where $$a_ia_k+b_ib_k+c_ic_k = \left[\begin{array}{l} 1,\ \text{if } i=k \\ 0,\ \text{if } i\neq k \end{array}\right. \qquad D = +1$$

Since there is one fixed point (*viz.*, the origin), there must be other fixed points. For the sake of simplicity, suppose that a point on the z-axis $(0, 0, c)$ remains fixed. Then

$$c_1 = c_2 = 0, \quad \text{so} \quad c_3 = +1$$

Therefore

$$a_3 = b_3 = 0$$

The transformation (18.5) can now be written as

$$\begin{aligned} x' &= a_1x + b_1y \\ y' &= a_2x + b_2y \\ z' &= z \end{aligned}$$

Since

$$a_1^2 + b_1^2 = a_2^2 + b_2^2 = 1$$

and

$$D = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 1,$$

we may put

$$a_1 = b_2 = \cos\theta, \qquad b_1 = -a_2 = \sin\theta$$

The transformation (18.5) can therefore be finally written as

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \qquad (18.5') \\ z' &= z \end{aligned}$$

The transformation (18.5′) shows that every point of the z-axis remains fixed and that there is rotation to the same amount in each of the planes $z = k$, for all values of k, about the point of intersection of that plane and the z-axis. In other words, every point, other than points on the z-axis, rotate about the z-axis in a plane perpendicular to the z-axis. The transformation is accordingly a rotation about the z-axis.

A rigid motion in space which leaves all points of a straight line fixed is called a *rotation about a fixed line*; the fixed line is called the *axis* of rotation. In a rotation about an axis g, all planes perpendicular to g are converted into themselves, although the individual points of any such plane, except the point of g, do not remain fixed.

A rigid motion may therefore be either a parallel displacement or a rotation about an axis or the identity.

2. *Case of symmetry*, $D = -1$.

Here

$$a_1 = -A_1, \quad b_2 = -B_2, \quad c_3 = -C_3$$

So, from (18.3),

$$\Delta = -2 + 2(a_1 + b_2 + c_3)$$

Hence, there is one fixed point if $a_1+b_2+c_3 \neq 1$ and there is either no fixed point or an infinity of fixed points if $a_1+b_2+c_3 = 1$

(*i*) Let $$a_1+b_2+c_3 = 1$$

If, for the sake of simplicity the origin and another point on the z-axis are supposed to remain fixed, then the transformation (18.1) may, as before, be written as

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= x\sin\theta - y\cos\theta \\ z' &= z \end{aligned} \tag{18.6}$$

This transformation shows that there is symmetry in each of the planes $z = k$, for all values of k. These planes are transformed into themselves and every point on the z-axis remains fixed.

If, moreover, $\cos\theta = -1$ and so $\sin\theta = 0$, we obtain

$$x' = -x, \quad y' = y, \quad z' = z \tag{18.6$'$}$$

By this transformation every point of the (y, z)-plane remains fixed.

Again the transformation (18.1) may be expressed as the product of the two transformations

$$\begin{aligned} \bar{x} &= a_1x+b_1y+c_1z \\ \bar{y} &= a_2x+b_2y+c_2z \\ \bar{z} &= a_3x+b_3y+c_3z \end{aligned} \qquad \text{and} \qquad \begin{aligned} x' &= \bar{x}+d_1 \\ y' &= \bar{y}+d_2 \\ z' &= \bar{z}+d_3 \end{aligned}$$

On the previous assumptions regarding the first of these two transformations, it can be reduced to (18.6′). Hence the transformation (18.1) is reduced in this case to

$$\begin{aligned} x' &= -x+d_1 \\ y' &= y+d_2 \\ z' &= z+d_3 \end{aligned} \tag{18.7}$$

There is, in general, no fixed point in this transformation.

(*ii*) Let $$a_1+b_2+c_3 \neq 1.$$

Suppose, for the sake of simplicity, the origin is the only fixed point and a point $(0, 0, c)$ is transformed into the point $(0, 0, -c)$. Then the transformation (18.1) takes the form

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= -z \end{aligned} \tag{18.8}$$

This can be decomposed into

$$\begin{aligned} \bar{x} &= x\cos\theta + y\sin\theta & \qquad x' &= \bar{x} \\ \bar{y} &= -x\sin\theta + y\cos\theta \qquad \text{and} & y' &= \bar{y} \\ \bar{z} &= z & z' &= -\bar{z} \end{aligned}$$

Therefore the transformation (18.8) consists of a rotation about the z-axis followed by an *orthogonal reflexion* in the plane $z = 0$.

71.1. Geometrical properties of orthogonal transformations. Let an orthogonal transformation (18.1) transform a vector (a, b, c) into a vector (a', b', c'). Without loss of generality we may suppose that

$$\begin{aligned} a &= x_2 - x_1, \quad b = y_2 - y_1, \quad c = z_2 - z_1, \\ a' &= x'_2 - x'_1, \quad b' = y'_2 - y'_1, \quad c' = z'_2 - z'_1, \end{aligned}$$

where the points (x_i, y_i, z_i) are transformed to the points (x'_i, y'_i, z'_i). Then the transformation of the coordinates of the vector are given by

$$\begin{aligned} a' &= a_1 a + b_1 b + c_1 c \\ b' &= a_2 a + b_2 b + c_2 c \\ c' &= a_3 a + b_3 b + c_3 c \end{aligned}$$

Let now two vectors $\alpha_1 = (u_1, v_1, w_1)$, $\alpha_2 = (u_2, v_2, w_2)$ be transformed by (18.1) into the vectors α'_1, α'_2 respectively. It follows from above and (18.1′) that the scalar product

$$\begin{aligned} \alpha'_1 . \alpha'_2 &= (a_1 u_1 + b_1 v_1 + c_1 w_1)(a_1 u_2 + b_1 v_2 + c_1 w_2) \\ &\quad + (a_2 u_1 + b_2 v_1 + c_2 w_1)(a_2 u_2 + b_2 v_2 + c_2 w_2) \\ &\quad + (a_3 u_1 + b_3 v_1 + c_3 w_1)(a_3 u_2 + b_3 v_2 + c_3 w_2) \\ &= u_1 u_2 + v_1 v_2 + w_1 w_2 = \alpha_1 . \alpha_2 \end{aligned}$$

Hence the scalar product remains invariant. If, in particular, $\alpha_1 = \alpha_2 = \alpha$, then $\alpha'_1 = \alpha'_2 = \alpha$. Therefore $|\alpha'|^2 = |\alpha|^2$. Hence the length of a vector is invariant. Therefore the absolute value of an angle remains invariant. In fact, it can be seen that an angle θ is transformed into an angle θ by rigid motion and into an angle $-\theta$ by symmetry.

On the other hand, if $P_1P_2P_3$ is a triangle,

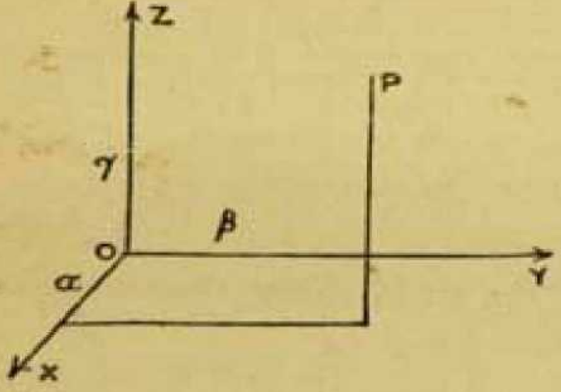
$$2\,|P_1P_2|\;|P_1P_3|\cos(P_1P_2, P_1P_3) = |P_2P_3|^2 - |P_1P_2|^2 - |P_1P_3|^2$$

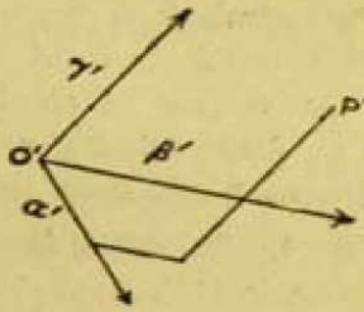
So, if the distance is invariant, the scalar product, the magnitude of angle and orthogonality remain invariant.

Thus, *a one-to-one correspondence between points for which the distance is an invariant is an orthogonal transformation.*

Finally, since the resultant of two orthogonal transformations is an orthogonal transformation, the above properties hold for successive applications of these transformations.

72. Affine transformations. A transformation which establishes a one-to-one correspondence between the points of the space and which transforms a vector into a vector, λ times a vector into λ times the transformed vector (λ being an arbitrary real number), is an *affine transformation*, or simply an affinity, of the space.





Let α, β, γ be the unit vectors in the directions of the positive x-, y-, z-axes respectively and let P be a point with coordinates (x, y, z) and O the origin. Then

$$\overline{OP} = x\alpha + y\beta + z\gamma \tag{18.9}$$

Let the points O, P and the vectors α, β, γ be transformed by an affinity to the points O', P' and the vectors α', β', γ' respectively. Then, since

$$(x\alpha, \quad y\beta, \quad z\gamma) \rightarrow (x\alpha', \quad y\beta', \quad z\gamma'),$$

we get

$$\overline{O'P'} = x\alpha' + y\beta' + z\gamma' \tag{18.9'}$$

Let the coordinates of $O', P', \alpha', \beta', \gamma'$ be (a_1, a_2, a_3), (x', y', z'), (a_{11}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{13}, a_{23}, a_{33}) respectively. Then, substituting these coordinates in (18.9') and arranging, we obtain

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_1 \\ y' &= a_{21}x + a_{22}y + a_{23}z + a_2 \\ z' &= a_{31}x + a_{32}y + a_{33}z + a_3 \end{aligned} \tag{18.10}$$

Since the vectors α, β, γ are linearly independent, the vectors α', β', γ' must also be linearly independent ; hence

$$| a_{ik} | \neq 0 \tag{18.10'}$$

Thus, the transformations (18.10) with (18.10′) are the *affinities* of the space. If a vector (a, b, c) is transformed by (18.10) into a vector (a', b', c'), then

$$\begin{aligned} a' &= a_{11}a + a_{12}b + a_{13}c \\ b' &= a_{21}a + a_{22}b + a_{23}c \\ c' &= a_{31}a + a_{32}b + a_{33}c \end{aligned} \tag{18.11}$$

Since λ times a vector is transformed into λ times the transformed vector, parallel vectors are transformed into parallel vectors with the ratio of the vectors invariant.

The tetrahedron whose coterminous edges are α, β, γ is transformed into the tetrahedron whose coterminous edges are α', β', γ'. But six times the volume of the latter tetrahedron is the determinant $|\,a_{ik}\,| \neq 0$. Hence, *the four vertices of a tetrahedron are transformed by an affinity into the four vertices of another tetrahedron.*

Let A_{ik} be the cofactors of a_{ik} in $|\,a_{ik}\,|$. Then the inverse of the transformation (18.10) is given by

$$\begin{aligned} x &= \frac{1}{|\,a_{ik}\,|}\Big[A_{11}x' + A_{21}y' + A_{31}z' - \Sigma a_i A_{i_1} \Big] \\ y &= \frac{1}{|\,a_{ik}\,|}\Big[A_{12}x' + A_{22}y' + A_{32}z' - \Sigma a_i A_{i_2} \Big] \\ z &= \frac{1}{|\,a_{ik}\,|}\Big[A_{13}x' + A_{23}y' + A_{33}z' - \Sigma a_i A_{i_3} \Big] \end{aligned} \tag{18.12}$$

Let a plane $u_1x + u_2y + u_3z + u_0 = 0$ be transformed by the affinity (18.10) into the plane $u_1'x' + u_2'y' + u_3'z' + u_0' = 0$. Then, by (18.12), it is seen that

$$\begin{aligned} \rho u_1' &= A_{11}u_1 + A_{12}u_2 + A_{13}u_3 \\ \rho u_2' &= A_{21}u_1 + A_{22}u_2 + A_{23}u_3 \\ \rho u_3' &= A_{31}u_1 + A_{32}u_2 + A_{33}u_3 \\ \rho u_0 &= -(u_1 \Sigma a_i A_{i_1} + u_2 \Sigma a_i A_{i_2} + u_3 \Sigma a_i A_{i_3}) + u_0\,|\,a_{ik}\,| \end{aligned}$$

This gives the transformation

$$(u_1, u_2, u_3, u_4) \rightarrow \rho(u_1', u_2', u_3', u_4'), \quad \rho \text{ being arbitrary}$$

Now let (18.10) be a given affinity by which we suppose, for the sake of simplicity, that the origin is left fixed; so

$$a_1 = a_2 = a_3 = 0$$

Obviously it transforms planes into planes and vectors into vectors.

A question that arises is this : Does there exist a plane $\epsilon \equiv u_1x + u_2y + u_3z = 0$ such that when $\epsilon = 0$ is transformed into the plane $\epsilon' \equiv u_1'x' + u_2'y' + u_3'z' = 0$ by the given affinity, the vector (u_1, u_2, u_3) normal to $\epsilon = 0$ is transformed into the vector (u_1', u_2', u_3') normal to $\epsilon' = 0$ at the same time ?

From the given transformation we have

$$\epsilon' = u_1'(a_{11}x + a_{12}y + a_{13}z) + u_2'(a_{21}x + a_{22}y + a_{23}z) + u_3'(a_{31}x + a_{32}y + a_{33}z)$$

Therefore

$$\rho u_1 = \Sigma a_{k1} u'_k, \quad \rho u_2 = \Sigma a_{k2} u'_k, \quad \rho u_3 = \Sigma a_{k3} u'_k,$$

where ρ is as yet arbitrary. But, by (18.11), the condition of the problem leads to

$$u_k' = \sum_j a_{kj} u_j \,; \quad \text{so, } \rho u_i = \sum_{j,k} a_{ki} a_{kj} u_j \quad j,k = 1, 2, 3$$

Put

$$c_{ij} = \sum_k a_{ki} a_{kj} \,; \quad \text{and so} \quad c_{ij} = c_{ji}, \quad | c_{ij} | \neq 0$$

Thus we have finally

$$\rho u_1 = c_{11}u_1 + c_{12}u_2 + c_{13}u_3$$
$$\rho u_2 = c_{21}u_1 + c_{22}u_2 + c_{23}u_3$$
$$\rho u_3 = c_{31}u_1 + c_{32}u_2 + c_{33}u_3$$

Solution of these three equations in u_1, u_2, u_3 other than all zero, exist if the determinant of the coefficients vanishes. That is, a plane $\epsilon = 0$ exists if

$$\begin{vmatrix} c_{11} - \rho & c_{12} & c_{13} \\ c_{21} & c_{22} - \rho & c_{23} \\ c_{31} & c_{32} & c_{33} - \rho \end{vmatrix} = 0$$

This is a cubic equation in ρ and so at least one of the roots of this equation is real. As a matter of fact, since $c_{ij} = c_{ji}$, it is proved in treatises on algebra that all the roots of such an equation are real. Hence, there exists plane $\epsilon = 0$ satisfying the given condition and the question is answered in the affirmative.

Now any vector which is parallel to the plane $\epsilon = 0$ is orthogonal to the vector (u_1, u_2, u_3) which is normal to the plane. Therefore it is possible to determine a pair of orthogonal vectors (or orthogonal lines) which are transformed into another pair of orthogonal vectors (or orthogonal lines) by a given affinity (as in § 23.2). This statement remains true when the origin is also transformed. It is, of course, evident that an arbitrary pair of orthogonal vectors do not transform in such a manner.

The next question that arises is whether there exists a system of three mutually orthogonal vectors which is transformed into another system of three mutually orthogonal vectors by a given affinity.

As a particular case, let the given affinity be

$$\begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \\ z' &= a_{33}z \end{aligned}$$

in which the (x, y)-plane and the z-axis are converted into themselves. If possible, let there exist a line $u_1x+u_2y = 0$ in the (x, y)-plane such that when this line is transformed into the line $u_1'x'+u_2'y' = 0$ by the given affinity, the vector (u_1, u_2) is transformed into the vector (u_1', u_2') at the same time. Then, from the given affinity, we get

$$u_1x+u_2y = u_1'(a_{11}x+a_{12}y)+u_2'(a_{21}x+a_{22}y)$$

Therefore

$$\sigma u_1 = a_{11}u_1'+a_{21}u_2', \quad \sigma u_2 = a_{12}u_1'+a_{22}u_2',$$

where σ is as yet unknown. But by hypothesis we have, from (18.11),

$$u_k' = \sum_j a_{kj}u_j; \quad \text{so} \quad \sigma u_i = \sum_{j,k} a_{ki}a_{kj}u_j, \quad j, k = 1, 2$$

Put

$$d_{ij} = \sum_k a_{ki}a_{kj}; \quad \text{and so} \quad d_{ij} = d_{ji}, \quad | d_{ij} | \neq 0$$

Thus finally

$$\begin{aligned} \sigma u_1 &= d_{11}u_1 + d_{12}u_2 \\ \sigma u_2 &= d_{21}u_1 + d_{22}u_2 \end{aligned}$$

Solution of these two equations in u_1, u_2, other than both zero, exists if the determinant of the coefficients vanishes. That is, a line $u_1x+u_2y = 0$ in the (x, y)-plane with the supposed property exists if

$$\begin{vmatrix} d_{11}-\sigma & d_{12} \\ d_{21} & d_{22}-\sigma \end{vmatrix} = 0$$

This is a quadratic equation in σ, the discriminant of which is

$$(d_{11}-d_{22})^2+4d_{12}^2$$

Since the discriminant is positive, the values of σ are real. Hence it is possible to determine the assumed line satisfying the given condition.

Now, any vector parallel to this line is orthogonal to the vector $(u_1, u_2, 0)$ and both these vectors are orthogonal to a vector parallel to the z-axis. Thus, in this particular case, the orthogonality of a certain system of three vectors remains unaltered.

Combining this with the previous result that the orthogonality of certain planes and their normals remains unaltered, we may state, in the case of general affinity, the following theorem :

Theorem. Every affinity transforms a suitable system of three orthogonal vectors into another system of three orthogonal vectors.

Consider an affinity given by

$$\begin{aligned} x' &= ax \\ y' &= by \qquad abc \neq 0 \\ z' &= cz \end{aligned} \tag{18.13}$$

or given by the product of a rigid motion and (18.13). If a, b, c are all different, there is only one system of orthogonal directions which is transformed into an orthogonal system. If two of the quantities a, b, c are equal, then in all systems of orthogonal directions which are transformed into orthogonal systems, one direction is uniquely defined and the other two form an arbitrary pair of orthogonal directions. If $a = b = c$, the affinity is a similarity (see § 57) and every system of three orthogonal vectors is transformed into an orthogonal system.

(The results obtained in the last four articles may be compared with the analogous results obtained in the plane geometry).

CHAPTER XIX

QUADRICS IN EUCLIDEAN SPACE

73. Pole, Polar. Tangent. The general equation of the second degree in the non-homogeneous coordinates can be written as

$$F(x, y, z) \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$$
$$+ 2a_1x + 2a_2y + 2a_3z + a = 0, \qquad (19.1)$$

where the coefficients of the second degree terms are not all zero. All surfaces satisfying this equation are surfaces of the second degree or quadrics.

In order to save space and also for the sake of convenience of notation, we shall write ξ_1, ξ_2, ξ_3 for x, y, z respectively, but shall go back to the x, y, z system of notation when not much advantage is gained by retaining the ξ's. Thus, the above equation can be written as

$$F(\xi_1, \xi_2, \xi_3) \equiv \sum_{i,k} a_{ik}\xi_i\xi_k + 2\sum_i a_i\xi_i + a = 0, \quad a_{ik} = a_{ki} \qquad (19.2)$$

Let $P = (\xi_1', \xi_2', \xi_3')$ be a given point which is neither on the quadric nor its centre (if the quadric is a central quadric). Take a line through P so as to meet the quadric in two points P_1, P_2 and let (p_1, p_2, p_3) be a vector parallel to this line. Then the coordinates of any point of the line will be given by (see (12.7))

$$\xi_i = \xi_i' + \rho p_i, \qquad i = 1, 2, 3 \qquad (19.3)$$

The directed segments $\overline{PP}_1, \overline{PP}_2$ will then be determined by the roots of

$$\sum a_{ik}(\xi_i' + \rho p_i)(\xi_k' + \rho p_k) + 2\sum a_i(\xi_i' + \rho p_i) + a = 0,$$

or

$$\rho^2\left(\sum a_{ik}p_ip_k\right) + 2\rho\left(\sum a_{ik}\xi_i'p_k\right) + \sum a_ip_i\right) + F(\xi_1', \xi_2', \xi_3') = 0$$

If ρ_1 and ρ_2 are the two roots of this equation,

$$\frac{2\rho_1\rho_2}{\rho_1 + \rho_2} = -\frac{F(\xi_1', \xi_2', \xi_3')}{\Sigma a_{ik}\xi_i'p_k + \Sigma a_ip_i}$$

If P' is the point on the line such that the cross-ratio $(PP', P_1P_2) = -1$, *i.e.*, the four points are harmonic, then

$$\overline{PP'} = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2}$$

Therefore, the coordinates of P' will be given by

$$\xi_i = \xi_i' - \frac{p_i F(\xi_1', \xi_2', \xi_3')}{\Sigma a_{ik}\xi_i'p_k + \Sigma a_ip_i}, \qquad i = 1, 2, 3$$

These are three equations between which and (19.3) p_1, p_2, p_3 can be eliminated. The eliminant is

$$\sum a_{ik}\xi_i'(\xi_k-\xi_k') + \sum a_i(\xi_i-\xi_i') + F(\xi_1', \xi_2', \xi_3') = 0$$

Finally this can be written, by virtue of the given equation of the quadric, as

$$\sum a_{ik}\xi_i\xi_k' + \sum a_i(\xi_i+\xi_i') + a = 0 \tag{19.4}$$

Since the point $P = (\xi_1', \xi_2', \xi_3')$ is given, the equation (19.4) is linear and so represents a plane. This plane is called the *polar plane* of the point P with respect to the given quadric. If the point P lies on the quadric, the equation (19.4) represents the *tangent plane* to the quadric at the point P.

If the polar plane of P intersects the quadric, it must intersect the surface in a conic. The lines joining P to the points of this conic are then tangents to the quadric at the points of the conic. These tangents therefore lie on a cone, called a *tangent cone* of the quadric.

All such properties can be worked out analytically exactly as in the plane geometry and it is needless to do so here.

74. Transformation of the general eqaution. Take a quadric given by the general equation (19.2), namely

$$F(\xi_1, \xi_2, \xi_3) \equiv \sum_{i,k} a_{ik}\xi_i\xi_k + 2\sum_i a_i\xi_i + a = 0, \qquad a_{ik} = a_{ki}$$

The right-hand expression may be considered as consisting of two parts, the (homogeneous) quadratic part $Q(\xi_1, \xi_2, \xi_3)$ and the linear part $l(\xi_1, \xi_2, \xi_3)$, where

$$Q(\xi_1, \xi_2, \xi_3) = \sum a_{ik}\xi_i\xi_k,$$
$$l(\xi_1, \xi_2, \xi_3) = \sum a_i\xi_i + a$$

Now apply a parallel displacement

$$\xi_i = \xi'_i - p_i, \qquad i = 1, 2, 3$$

Then (19.2) is transformed as

$$\sum_{i,k} a_{ik}(\xi'_i - p_i)(\xi'_k - p_k) + 2\sum_i a_i(\xi'_i - p_i) + a = 0,$$

or

$$\sum_{i,k} b_{ik}\xi'_i\xi'_k + 2\sum_i b_i\xi'_i + b = 0, \tag{19.5}$$

where

$$b_{ik} = a_{ik}, \qquad b_i = a_i - \sum a_{ik}p_k, \qquad b = \sum a_{ik}p_ip_k - 2\sum a_ip_i + a$$

If the quadric has a centre and new origin $(-p_1, -p_2, -p_3)$ is chosen as the centre, then the transformed equation (19.3) must be satisfied by $(-\xi'_1, -\xi'_2, -\xi'_3)$. Hence

$$b_1 = b_2 = b_3 = 0$$

That is (p_1, p_2, p_3) must be the solution of the three equations

$$\sum a_{ik}p_k - a_i = 0, \qquad i = 1, 2, 3$$

The necessary and sufficient condition for this is that the determinant $| a_{ik} | \neq 0$, *i.e.*, the rank of (a_{ik}) is three. If, however, the rank of (a_{ik}) be less than three, then either there is no centre or there are an infinity of centres.

Thus, *if the rank of* (a_{ik}) *is three, there exists one centre and the general eqution (19.2) can be transformed into the form*

$$\sum b_{ik}\xi'_i\xi'_k + b = 0 \tag{19.6}$$

Now consider the case when the rank of (a_{ik}) is less than three, *i.e.*, $| a_{ik} | = 0$. In this case it is possible to choose three numbers e_{31}, e_{32}, e_{33}, not all zero, such that the three equations

$$\sum a_{ik}e_{3k} = 0, \qquad i = 1, 2, 3, \tag{19.7}$$

are satisfied. Let these numbers be so chosen that

$$e_{31}{}^2 + e_{32}{}^2 + e_{33}{}^2 = 1,$$

i.e., (e_{3i}) is a unit vector. We take two other unit vectors (e_{1i}), (e_{2i}) so that the three vectors form an orthogonal system.

Now apply the orthogonal transformation (it may be a rigid motion if the unit vectors are so chosen)

$$\xi_i = \sum_{\mu} e_{\mu i}\xi'_{\mu}, \qquad i = 1, 2, 3$$

Then the quadratic part of (19.2) is transformed into

$$\sum_{i,k} a_{ik}\Big(\sum_{\mu} e_{\mu i}\xi'_{\mu}\Big)\Big(\sum_{\nu} e_{\nu k}\xi'_{\nu}\Big) = \sum_{i,k,\mu,\nu} a_{ik}e_{\mu i}e_{\nu k}\xi'_{\mu}\xi'_{\nu} = \sum_{\mu,\nu} c_{\mu\nu}\xi'_{\mu}\xi'_{\nu}, \text{ say,}$$

where

$$c_{\mu\nu} = \sum_{i,k} a_{ik}e_{\mu i}e_{\nu k}\,; \quad \text{and so} \quad c_{\mu\nu} = c_{\nu\mu}$$

So, by (19.7),

$$c_{\mu 3} = \sum_{i}\Big(\sum_{k} a_{ik}e_{3k}\Big)e_{\mu i} = 0, \qquad \mu = 1, 2, 3$$

Therefore ultimately

$$\sum a_{ik}\xi_i\xi_k \rightarrow c_{11}\xi'_1{}^2 + 2c_{12}\xi'_1\xi'_2 + c_{22}\xi'_2{}^2 \tag{19.8}$$

Now the rank of (c_{ik}) is equal to the rank of (a_{ik}), because the rank is unaltered by the linear transformation applied. So if the rank of (a_{ik}) is two, then the rank of (c_{ik}) is also two.

Thus, *if the rank of* (a_{ik}) *is two, the quadratic part of (19.2) can be transformed by suitable orthogonal transformation into a quadratic function of two variables, as in (19.8).*

Finally suppose that the rank of (a_{ik}) is one. Take the quadratic (19.8) in two variables, namely

$$Q'(\xi_1', \xi_2') \equiv c_{11}\xi_1'^2 + 2c_{12}\xi_1'\xi_2' + c_{22}\xi_2'^2$$

As the rank is less than two, $c_{11}c_{22} - c_{12}^2 = 0$. So, two numbers δ_1, δ_2 may be so chosen that, for an arbitrary factor k,

$$c_{11} = k\delta_1^2, \quad c_{12} = k\delta_1\delta_2, \quad c_{22} = k\delta_2^2,$$

i.e.,

$$c_{ik} = k\delta_i\delta_k\,;$$

and as the rank of (c_{ik}) is one, $k \neq 0$. We then have

$$Q'(\xi_1', \xi_2') = k\,[\delta_1^2\xi_1'^2 + 2\,\delta_1\delta_2\xi_1'\xi_2' + \delta_2^2\xi_2'^2]$$

Choose δ_1, δ_2 so thet $\delta_1^2 + \delta_2^2 = 1$ and apply the orthogonal transformation (which may be chosen as a rigid motion)

$$\begin{aligned}\xi_1' &= \delta_1\xi_1'' - \delta_2\xi_2''\\ \xi_2' &= \delta_2\xi_1'' + \delta_1\xi_2''\\ \xi_3' &= \xi_3''\end{aligned}$$

So $Q'(\xi_1', \xi_2')$ is transformed into

$$\begin{aligned}k\,[\delta_1^2(\delta_1\xi'' - \delta_2\xi_2'')^2 + 2\,\delta_1\delta_2(\delta_1\xi_1'' - \delta_2\xi_2'')\,(\delta_2\xi_1'' + \delta_1\xi_2'')\\ + \delta_2^2(\delta_2\xi_1'' + \delta_1\xi_2'')^2] = k\xi_1''^2 \qquad (19.9)\end{aligned}$$

Thus, *when the rank of* (a_{ik}) *is one, the quadratic part of (19.2) can be transformed by suitable orthogonal transformation into a quadratic function of one variable, as in (19.9).*

75. Metric Classification of quadrics. We start with the general equation (19.1), namely

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_1x + 2a_2y + 2a_3z + f = 0,$$

where the coefficients of the second degree terms are not all zero.

We have to consider the different cases that arise according as the rank of matrix (a_{ik}) is one, two or three.

I. Let the rank of (a_{ik}) be one. Then, by (19.9) the general equation can be transformed into the form

$$x^2 + 2ax + 2by + 2cz + d = 0$$

Now apply the parallel displacement

$$x' = x + a$$
$$y' = y$$
$$z' = z$$

Then the equation reduces to

$$x'^2 + 2by' + 2cz' + d' = 0, \quad \text{where} \quad d' = d - a^2$$

(*i*) If b, c are both zero, the equation reduces to (writing x for x')

$$x^2 + d' = 0$$

There are here three cases according as $d' \lesseqgtr 0$. Correspondingly we obtain the following *normal forms* :

$$x^2 = k^2, \tag{19.10}$$

representing *a pair of parallel planes.*

$$x^2 = 0, \tag{19.11}$$

representing *a pair of coincident planes.*

$$x^2 = -k^2, \tag{19.12}$$

representing *a pair of parallel planes without real trace.*

(*ii*) If b, c are not both zero, apply the rigid motion

$$x'' = x'$$
$$y'' = -\frac{1}{p}(by' + cz') - \frac{d'}{2p} \qquad p = |\sqrt{b^2 + c^2}|$$
$$z'' = -\frac{1}{p}(-cy' + bz')$$

Then the equation reduces to (dropping the dashes)

$$x^2 - 2py = 0, \tag{19.13}$$

representing *a parabolic cylinder.*

II. Let the rank of (a_{ik}) be two and let

$$a_{11}a_{22} - a_{12}^2 \neq 0$$

Then, by (19.8), the general equation can be transformed into the form

$$g_{11}x^2 + 2g_{12}xy + g_{22}y^2 + 2(g_1x + g_2y + g_3z) + g = 0,$$

where

$$g_{11}g_{22} - g_{12}^2 \neq 0$$

Apply the parallel displacement

$$x' = x - c_1$$
$$y' = y - c_2$$
$$z' = z$$

The equation now takes the form

$$g_{11}x'^2 + 2g_{12}x'y' + g_{22}y'^2 + 2(a'x' + b'y' + c'z') + g' = 0,$$

where

$$a' = g_{11}c_1 + g_{12}c_2 + g_1,$$

$$b = g_{12}c_1 + g_{22}c_2 + g_2$$

Choose c_1 and c_2 such that $a' = b' = 0$; this is possible because $g_{11}g_{22} \neq g_{12}^2$. So the equation reduces to

$$g_{11}x'^2 + 2g_{12}x'y' + g_{22}y'^2 + 2c'z' + g' = 0$$

(*i*), (*ii*) If $c' = 0$, two cases may arise according as $g' = 0$ or $g' \neq 0$. In the latter case, we may, without loss of generality, suppose $g' = -1$.

(*iii*) If $c' \neq 0$, we may, without loss of generality, suppose $2c' = -1$; and then write z' for $z' - g'$ (*i.e.*, apply a parallel displacement).

In these three cases, the equation takes the forms

$$g_{11}x'^2 + 2g_{12}x'y' + g_{22}y'^2 = \left[\begin{array}{l} 0 \\ 1 \\ z' \end{array}\right. \qquad (A)$$

Now apply a rotation about the z'-axis

$$x'' = x' \cos\theta - y' \sin\theta$$

$$y'' = x' \sin\theta + y' \cos\theta$$

$$z'' = z'$$

and choose θ so that the coefficient of $x''y''$ in the equations to which the three equations (A) are transformed vanishes. The coefficient of $x''y''$ in the transformed equations is

$$(g_{11} - g_{22}) \sin 2\theta + 2g_{12} \cos 2\theta$$

This can vanish if $\tan 2\theta = 2g_{12}/(g_{22} - g_{11})$. Hence finally the equations (A) reduce to the forms

$$\pm \frac{x''^2}{a^2} \pm \frac{y''^2}{b^2} = \left[\begin{array}{l} 0 \\ 1 \\ z'' \end{array}\right. \qquad (B)$$

The different cases that may arise from the equations (B) are (dropping the dashes) given by the following *normal forms*:

$$x^2/a^2 + y^2/b^2 = 0, \qquad (19\ 14)$$

representing *a pair of planes without real trace but intersecting in a real line.*

$$x^2/a^2 - y^2/b^2 = 0, \qquad (19.15)$$

representing *a pair of planes.*

$$x^2/a^2 + y^2/b^2 = 1, \qquad (19.16)$$

representing *an elliptic cylinder.*

$$-x^2/a^2 - y^2/b^2 = 1, \qquad (19.17)$$

representing *an elliptic cylinder without real trace.*

$$x^2/a^2 - y^2/b^2 = 1, \qquad (19.18)$$

representing *a hyperbolic cylinder.*

$$x^2/a^2 + y^2/b^2 = z, \qquad (19.19)$$

representing *an elliptic paraboloid.* The surface (19.19) may by generated by a variable ellipse

$$x^2/a^2 + y^2/b^2 = k, \qquad z = k$$

The ellipse is without real trace if $k < 0$, real if $k > 0$ and consists of a point (the origin) if $k = 0$; also the sections of the surface by the planes $x = 0$, $y = 0$ are parabolas. To resume, we have the further *normal form*

$$x^2/a^2 - y^2/b^2 = z, \qquad (19.20)$$

representing *a hyperbolic paraboloid.* The surface (19.20) may be generated by a variable hyperbola

$$x^2/a^2 - y^2/b^2 = k, \qquad z = k$$

The hyperbola consists of a pair of lines if $k = 0$ and is real for real values of k, the centre being always on the z-axis; also the sections of the surface by the planes $x = 0$, $y = 0$ are parabolas.

The two equations (19.19) and (19.20) may be put compactly as

$$x^2/a^2 + \delta\, y^2/b^2 = z, \qquad \delta = \pm 1$$

Consider a straight line which is parallel to a vector (p, q, r) and whose points are given by the coordinates

$$(x_1 + \rho p, \quad y_1 + \rho q, \quad z_1 + \rho r),$$

where (x_1, y_1, z_1) is a point on a paraboloid. If this lies wholly on the paraboloid, we must have

$$(x_1 + \rho p)^2/a^2 + \delta(y_1 + \rho q)^2/b^2 = z_1 + \rho r,$$

or

$$\rho^2(p^2/a^2 + \delta q^2/b^2) + \rho(2x_1\, p/a^2 + 2\delta y_1\, q/b^2 - r) = 0$$

satisfied by all values of ρ. Hence

$$p^2/a^2 + \delta q^2/b^2 = 0, \quad 2x_1 p/a^2 + 2\delta y_1 q/b^2 - r = 0$$

Both these equations cannot evidently be satisfied by $\delta = +1$. So, no straight line can lie wholly on an elliptic paraboloid. If $\delta = -1$, the first of the above two equations gives

$$(p/a + q/b)(p/a - q/b) = 0; \quad \text{and so} \quad p/q = \pm a/b$$

Each of these two values of $p:q$ defines $p:q:r$ uniquely by virtue of the second equation.

Therefore, *through every point of a hyperbolic paraboloid there pass two straight lines lying wholly on the surface.* These lines are called the *generators* of the surface.

III. Let the rank of the matrix (a_{ik}) be three. Then the quadric has a single centre and, by (19.4), the general equation can be transformed into the form (writing ξ_1, ξ_2, ξ_3 for x, y, z)

$$\sum_{i,k} a_{ik}\xi_i\xi_k + d = 0, \quad a_{ik} = a_{ki}, \quad | a_{ik} | \neq 0$$

Apply the orthogonal transformation (it may be a rigid motion if the coefficients e_{ik} are so chosen)

$$\xi_i = \sum_{\mu} e_{i\mu}\eta_{\mu}, \qquad i = 1, 2, 3,$$

where (e_{11}, e_{21}, e_{31}), (e_{12}, e_{22}, e_{32}), (e_{13}, e_{23}, e_{33}) are mutually orthogonal vectors. This transformation transforms the equation into

$$\sum_{\mu,\nu} g_{\mu\nu}\eta_{\mu}\eta_{\nu} + d = 0, \tag{C}$$

where

$$g_{\mu\nu} = \sum_{i,k} a_{ik}e_{i\mu}e_{k\nu}; \quad \text{and so} \quad g_{\mu\nu} = g_{\nu\mu}$$

Therefore

$$g_{31} = \sum_{i,k} a_{ik}e_{i_3}e_{k_1} = \sum_{k} e_{k_1}(\sum_{i} a_{ik}e_{i_3})$$

$$g_{32} = \sum_{i,k} a_{ik}e_{i_3}e_{k_2} = \sum_{k} e_{k_2}(\sum_{i} a_{ik}e_{i_3})$$

It is now seen that g_{31} and g_{32} may be made to vanish if the vectors (e_{k_1}) and (e_{k_2}) are both made orthogonal to the vector $(\sum\limits_{i} a_{ik}e_{i_3})$, $k = 1, 2, 3$. In that case the vectors (e_{k_3}) and $(\sum\limits_{i} a_{ik}e_{i_3})$ have to be parallel. So let us put

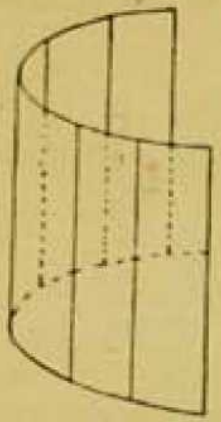
$$\rho e_{13} = \sum_{i} a_{i_1}e_{i_3}$$

$$\rho e_{23} = \sum_{i} a_{i_2}e_{i_3} \qquad \rho \neq 0$$

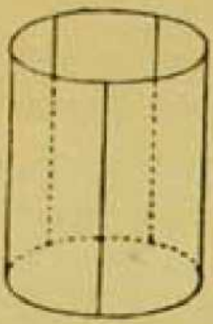
$$\rho e_{33} = \sum_{i} a_{i_3}e_{i_3}$$

There exists a solution of these equations in e_{13}, e_{23}, e_{33} if

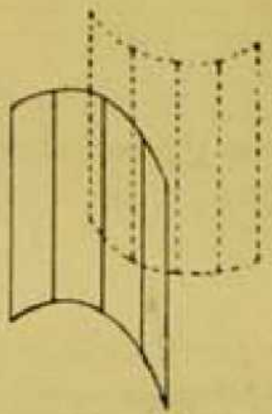
$$\begin{vmatrix} a_{11}-\rho & a_{12} & a_{13} \\ a_{21} & a_{22}-\rho & a_{23} \\ a_{31} & a_{32} & a_{33}-\rho \end{vmatrix} = 0$$



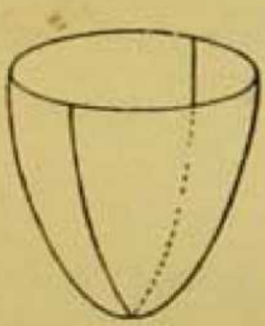
PARABOLIC CYLINDER



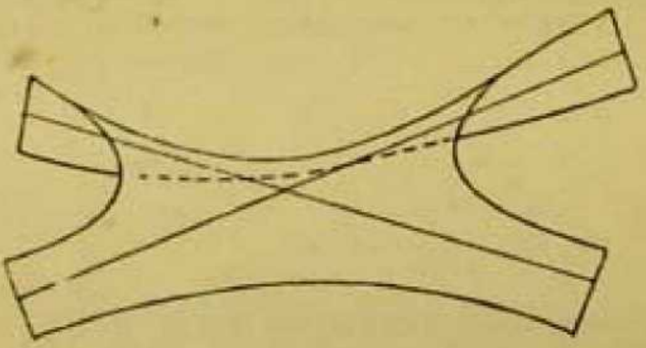
ELLIPTIC CYLINDER



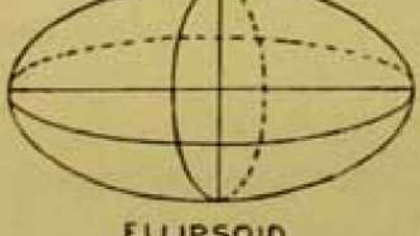
HYPERBOLIC CYLINDER



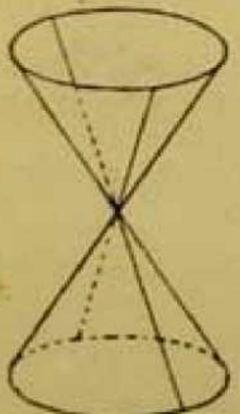
ELLIPTIC PARABOLOID



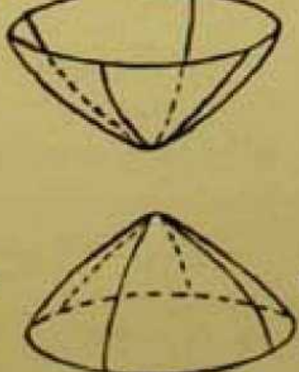
HYPERBOLIC PARABOLOID



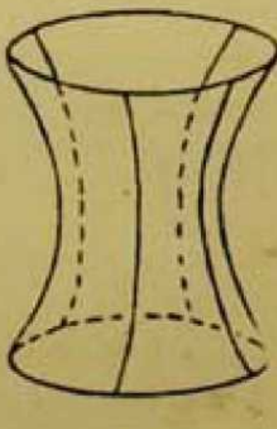
ELLIPSOID



CONE OF THE SECOND
DEGREE



HYPERBOLOID OF TWO SHEETS



HYPERBOLOID OF ONE
SHEET

This is a cubic equation in ρ and so a real solution exists. Thus we see that g_{31} and g_{32} may be made to vanish by a suitable orthogonal transformation. Hence the equation (C) reduces to

$$g_{11}\eta_1^2+2g_{12}\eta_1\eta_2+g_{22}\eta_2^2+g_{33}\eta_3^2+d=0$$

Now as in the case II we may apply a suitable rotation

$$\begin{aligned}\zeta_1&=\eta_1\cos\theta-\eta_2\sin\theta\\ \zeta_2&=\eta_1\sin\theta+\eta_2\cos\theta\\ \zeta_3&=\eta_3\end{aligned}$$

and choose θ such that the coefficient of $\zeta_1\,\zeta_2$ in the transformed equation vanishes. When θ is so chosen the equation is transformed into the form

$$d_{11}\zeta_1^2+d_{22}\zeta_2^2+d_{33}\zeta_3^2+d=0$$

Finally, two cases may arise according as $d=0$ or $d\neq 0$. In the latter case we may, without loss of generality, assume $d=-1$.

In these two cases the general equation takes the forms (writing x, y, z for $\zeta_1, \zeta_2, \zeta_3$)

$$\pm\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=\left[\begin{matrix}0\\1\end{matrix}\right. \tag{D}$$

The different cases that may arise from the equations (D) are given by the followin *normal forms* :

$$x^2/a^2+y^2/b^2+z^2/c^2=0, \tag{19.21}$$

representing *a cone of the second degree without real trace but with a real vertex.*

$$x^2/a^2+y^2/b^2-z^2/c^2=0, \tag{19.22}$$

representing *a cone of the second degree* [a surface of this type has been considered under (15.15)]. We have further

$$x^2/a^2+y^2/b^2+z^2/c^2=1, \tag{19.23}$$

representing *an ellipsoid*. The surface (19.23) may be generated by a variable ellipse

$$x^2/a^2+y^2/b^2=1-k^2/c^2,\quad z=k,\quad -c\leq k\leq c$$

The surface is therefore bounded in every direction ; also the sections of the surface by the coordinate planes are ellipses. To resume, we have further

$$x^2/a^2+y^2/b^2-z^2/c^2=1, \tag{19.24}$$

representing *a hyperboloid of one sheet*. The surface (19.24) may be generated by a variable ellipse

$$x^2/a^2+y^2/b^2=1+k^2/c^2,\quad z=k$$

Also the sections of the surface by the planes $x = 0$, $y = 0$ are hyperbolas. To proceed, we have further

$$-x^2/a^2 - y^2/b^2 + z^2/c^2 = 1, \tag{19.25}$$

representing *a hyperboloid of two sheets*. The surface (19.25) may be generated by a variable ellipse

$$x^2/a^2 + y^2/b^2 = k^2/c^2 - 1, \quad z = k,$$

k not lying between $+c$ and $-c$; also the sections of the surface by the planes $x = 0$, $y = 0$ are hyperbolas. To resume, we have finally

$$-x^2/a^2 - y^2/b^2 - z^2/c^2 = 1, \tag{19.26}$$

representing *a surface without real trace*. We may now state the following :

The surfaces (19.10) to (19.26) exhaust all the different types or classes of quadric surfaces in the Euclidean space.

It may be seen, as in the case of a hyperbolic paraboloid, that there are straight lines lying wholly on a hyperboloid of one sheet. Let the points of a straight line be given by

$$(x_1 + \rho p, \quad y_1 + \rho q, \quad z_1 + \rho r),$$

where (x_1, y_1, z_1) is a point on the hyperboloid of one sheet (19.24). If the straight line lies wholly on the surface, we must have

$$(x_1 + \rho p)^2/a^2 + (y_1 + \rho q)^2/b^2 - (z_1 + \rho r)^2/c^2 = 1$$

satisfied by all values of ρ. Hence we have the three equations

$$\begin{aligned} x_1^2/a^2 + y_1^2/b^2 - 1 &= z_1^2/c^2 \\ x_1 p/a^2 + y_1 q/b^2 &= z_1 r/c^2 \\ p^2/a^2 + q^2/b^2 &= r^2/c^2 \end{aligned} \tag{19.27}$$

Eliminating $z_1 r$ between the equations, we get

$$(x_1^2/a^2 + y_1^2/b^2 - 1)(p^2/a^2 + q^2/b^2) = (x_1 p/a^2 + y_1 q/b^2)^2,$$

or

$$p^2(y_1^2 - b^2) + q^2(x_1^2 - a^2) - 2pqx_1y_1 = 0$$

The discriminant of this equation, considered as a quadratic in $p : q$, is

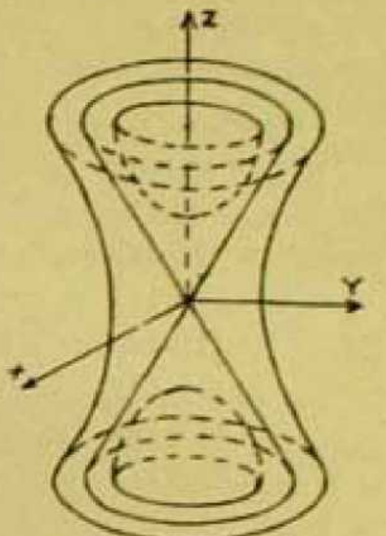
$$4\{x_1^2y_1^2 - (x_1^2 - a^2)(y_1^2 - b^2)\}$$
$$= 4a^2b^2(x_1^2/a^2 + y_1^2/b^2 - 1) = 4a^2b^2z_1^2/c^2$$

If $z_1 \neq 0$, the discriminant is positive and so there are two real values of $p : q$ and hence, by (19.27), there are two real values of $p : q : r$. If $z_1 = 0$, the discriminant vanishes and so there is one real solution $p : q$ and hence, by (19.27), two real values of $p : q : r$.

Hence, *through each point of a hyperboloid of one sheet there pass two lines lying wholly on the surface.* These lines are the *generators* of the surface. Since the generators satisfy the last two of the equations (19.27), the lines through the origin parallel to the generators lie on the cone (19.22), namely

$$x^2/a^2+y^2/b^2-z^2/c^2=0$$

This cone is called the *asymptotic cone* of the surface (19.24). In a similar manner, (19.22) is also the asymptotic cone of the surface (19.25).



The above three equations (19.24), (19.25) and (19.22) of the hyperboloids and their asymptotic cone can be put compactly as

$$x^2/a^2+y^2/b^2=\begin{cases}(z^2\pm c^2)/c^2\\ z^2/c^2\end{cases}$$

The sections of these surfaces by the planes $z=k$, for suitable k, are similar and similarly situated ellipses and they approximate one another as k tends to infinity. The section of the first surface by $z=k$ and of the cone by $z=\sqrt{k^2+c^2}$ are congruent ellipses.

76. Affine classification of quadrics. The seventeen different types of the second degree surfaces (19.10) to (19.26) that we have obtained in the last article are different from one another from the point of view of affine transformation in the sense that it is not possible to transform, by an affine transformation, any one of these seventeen surfaces to another. On the other hand, all surfaces of the same type are *equivalent* from the point of view of affine transformation though not from the point of view of orthogonal transformation. For example, given two parabolic cylinders [obtained by giving different values to p in (19.13)], one can be transformed into another by suitable affine transformation ; we express this by saying that all parabolic cylinders are *affine*. This is true for each of the seventeen types of surfaces. The following table shows the *affine normal forms* to which the equations (19.10) to (19.26) reduce when they are transformed by suitable affine transformations :

Equations (19.10) to (19.12) transform into $x^2=\begin{cases}1\\0\\-1\end{cases}$

(19.13) . . . $x^2-y=0$

(19.14) to (19.18) . . . $\pm x^2\pm y^2=\begin{cases}0\\1\end{cases}$ (19.28)

(19.19) and (19.20) . . . $x^2\pm y^2=z$

(19.21) to (19.26) . . . $\pm x^2\pm y^2\pm z^2=\begin{cases}0\\1\end{cases}$

In § 61, under projective classification of quadrics, we have eight different classes of quadrics (15.11) to (15.18). It therefore follows that some classes of quadrics which are different from the affine point of view must be considered as equivalent from the projective point of view. By writing down the equations (19.10) to (19.26) in homogeneous coordinates, the equivalence of the projective and affine classes of surfaces can be obtained as follows : (19.29)

Projective classes of surface	Equivalent to	Affine classes of surfaces
(15.11)	. . .	(19.11)
(15.12)	. . .	(19.12), (19.14)
(15.13)	. . .	(19.10), (19.15)
(15.14)	. . .	(19.17), (19.21)
(15.15)	. . .	(19.13), (19.16), (19.18), (19.22)
(15.16)	. . .	(19.26)
(15.17)	. . .	(19.20), (19.23), (19.25)
(15.18)	. . .	(19.19), (19.24)

77. Generators of the hyperboloid of one sheet and of the hyperbolic paraboloid. A parametric representation of (19.24) *i.e.*, of the hyperboloid of one sheet $x^2/a^2+y^2/b^2-z^2/c^2 = 1$, is given by

$$\begin{aligned} x &= \frac{a}{c}\,|\,\sqrt{t^2+c^2}\,|\,\cos\phi \\ y &= \frac{b}{c}\,|\,\sqrt{t^2+c^2}\,|\,\sin\phi \\ z &= t \end{aligned} \tag{19.30}$$

where t and ϕ are parameters ; and a parametric representation of the directions of the generators of the surface is given by

$$\begin{aligned} p &= \frac{a}{c}\,\mu\cos\psi \\ q &= \frac{b}{c}\,\mu\sin\psi \\ r &= \mu \end{aligned} \tag{19.31}$$

where μ and ψ are parameters. Since x, y, z and p, q, r satisfy the second of the equations (19.27), namely $xp/a^2+yq/b^2 = zr/c^2$, we have

$$|\,\sqrt{t^2+c^2}\,|\,(\cos\phi\cos\psi+\sin\phi\sin\psi) = t,$$

or

$$|\,\sqrt{t^2+c^2}\,|\,\cos(\psi\sim\phi) = t$$

So
$$\cos(\psi-\phi)=\frac{t}{|\sqrt{t^2+c^2}|},\qquad \sin(\psi-\phi)=\pm\frac{c}{|\sqrt{t^2+c^2}|}\tag{19.32}$$

Now the coordinates of any point of the orthogonal projection of any generator on the plane $z=0$ are, by (19.30) and (19.31),

$$x=\frac{a}{c}\,|\sqrt{t^2+c^2}|\cos\phi+\rho\frac{a}{c}\cos\psi$$

$$y=\frac{b}{c}\,|\sqrt{t^2+c^2}|\sin\phi+\rho\frac{b}{c}\sin\psi$$

Eliminating ρ between these equations, we get

$$bx\sin\psi-ay\cos\psi=\frac{ab}{c}\,|\sqrt{t^2+c^2}|\sin(\psi-\phi)$$

Therefore, by (19.32), the orthogonal projections of the generators on the plane $z=0$ are the lines

$$bx\sin\psi-ay\cos\psi=\pm ab\tag{19.33}$$

On the other hand, the section of the hyperboloid of one sheet by $z=0$ is the ellipse

$$x^2/a^2+y^2/b^2=1$$

and the condition that a line $u_1x+u_2y+u_3=0$ in the (x, y)-plane be tangent to this ellipse is that

$$u_1^2a^2+u_2^2b^2-u_3^2=0$$

This condition is identically satisfied for the lines (19.33). It therefore follows that *the orthogonal projections of the generators of (19.24) on the plane $z=0$ are tangents to the section of the surface by the same plane.*

We have seen that two generators pass through every point of the surface and therefore through every point of the section of the surface by $z=0$. The coordinates of these latter points are given, from (19.30), by

$$(a\cos\phi,\quad b\sin\phi,\quad 0).$$

Since $t=0$, we have, by (19.32),

$$\cos(\psi-\phi)=0\,;\quad\text{and so}\quad \psi-\phi=\pm\pi/2$$

We thus obtain *two systems of generators*, one for $\psi-\phi=+\pi/2$ and the other for $\psi-\phi=-\pi/2$.

For convenience, put $\mu=c$. Then the points of the generators of the two systems (through the points of the section of the surface by $z=0$) are given, from (19.30) and (19.31), by

$$\begin{aligned}x&=a\cos\phi+\rho\,a\cos(\phi\pm\pi/2)\\ y&=b\sin\phi+\rho\,b\sin(\phi\pm\pi/2)\\ z&=\rho\,c\end{aligned}$$

That is, the generators of the two systems are given by

$$\begin{aligned} x &= a\,(\cos\phi-\rho\sin\phi) & \qquad\qquad x &= a\,(\cos\phi+\rho\sin\phi) \\ y &= b\,(\sin\phi+\rho\cos\phi) & \text{and}\qquad y &= b\,(\sin\phi-\rho\cos\phi) \\ z &= c\,\rho & z &= c\,\rho \end{aligned} \tag{19.34}$$

Two generators of the same system are obtained by giving different values to the parameters (ρ, ϕ), say (ρ_1, ϕ_1) and (ρ_2, ϕ_2), in that system. If two generators of the same system, say the first of (19.34), have a point in common, then, for this common point, we must have

$$\rho_1 = \rho_2 = \rho$$

and

$$\cos\phi_1-\cos\phi_2 = \rho\,(\sin\phi_1-\sin\phi_2),$$

$$\sin\phi_1-\sin\phi_2 = -\rho\,(\cos\phi_1-\cos\phi_2)$$

Therefore $$1 = -\rho^2$$

This shows that there is no real value of ρ. Hence, *no two generators of the same system can intersect one another* ; they are skew lines.

Again, take two generators of the different systems :

$$\begin{aligned} x &= a\,(\cos\phi_1-\rho_1\sin\phi_1) & \qquad\qquad x &= a\,(\cos\phi_2+\rho_2\sin\phi_2) \\ y &= b\,(\sin\phi_1+\rho_1\cos\phi_1) & \text{and}\qquad y &= b\,(\sin\phi_2-\rho_2\cos\phi_2) \\ z &= c\,\rho_1 & z &= c\,\rho_2 \end{aligned}$$

If these two generators have a point in common, we must have

$$\rho_1 = \rho_2 = \rho$$

and

$$\cos\phi_1-\cos\phi_2 = \rho(\sin\phi_1+\sin\phi_2),$$

$$\sin\phi_1-\sin\phi_2 = -\rho(\cos\phi_1+\cos\phi_2)$$

Multiplying,

$$\sin(\phi_1+\phi_2)-\tfrac{1}{2}(\sin 2\phi_1+\sin 2\phi_2)$$

$$= \rho^2\{\sin(\phi_1+\phi_2)+\tfrac{1}{2}(\sin 2\phi_1+\sin 2\phi_2)\}$$

Therefore

$$\rho^2 = \frac{1-\cos(\phi_1-\phi_2)}{1+\cos(\phi_1-\phi_2)}$$

This shows that ρ is always real unless $\cos(\phi_1-\phi_2) = -1$, in which case the two generators are parallel. Hence, *two generators of the different systems are always coplanar ; they either intersect in a point or are parallel.*

Now consider the generators of (19.20), *i.e.*, of the hyperbolic paraboloid $x^2/a^2-y^2/b^2 = z$. The surface and the directions of its generators may be given parametrically by

$$x = at\cos\theta,\quad y = bt\sin\theta,\quad z = t^2\cos 2\theta$$

$$\text{and}\qquad p = a\mu,\qquad q = \pm b\mu,\qquad r = 2t\mu(\cos\theta\mp\sin\theta). \tag{19.35}$$

where t, θ, μ are parameters. The orthogonal projections of the generators on the plane $z = 0$ are lines given by

$$x = at\cos\theta + \rho\, a\mu$$

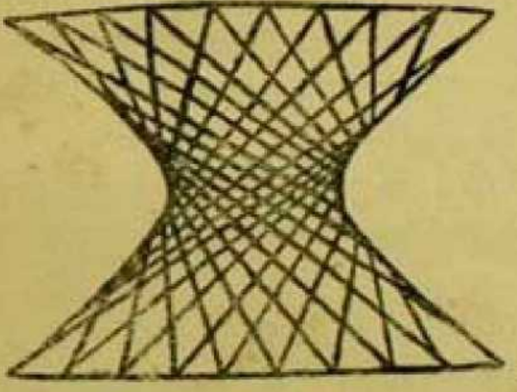
$$y = bt\sin\theta \pm \rho\, b\mu$$

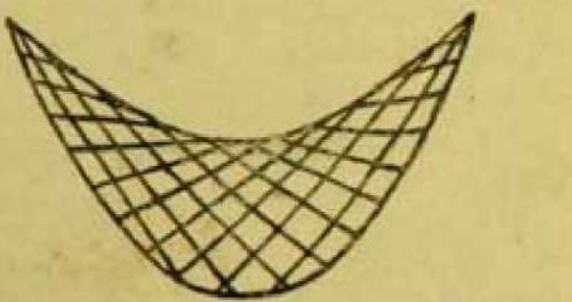
Eliminating ρ between these two sets of equations we obtain the two equations

$$bx - ay = \gamma, \quad bx + ay = \lambda,$$

where γ and λ are arbitrary constants. Thus, *the orthogonal projections of the generators of (19.20) on the plane $z = 0$ are two sets of parallel lines and every point of intersection of these lines is the projection of only one point of the surface.*

Exactly as in the case of generators of a hyperboloid of one sheet discussed above, it may be seen that *there are two systems of generators of a hyperbolic paraboloid; no two generators of the same system can intersect one another, and two generators of different systems either intersect or are parallel.* The generators of the two surfaces are shown in the diagrams given below :





78. Plane sections of quadrics. As in (19.2), take the general equation of a quadric in the current coordinates ξ_1, ξ_2, ξ_3 as

$$\sum_{i,k} a_{ik}\xi_i\xi_k + 2\sum_i a_i\xi_i + a = 0, \quad a_{ik} = a_{ki}, \quad i,k = 1, 2, 3 \tag{19.36}$$

Let ϵ be any plane which is supposed to intersect the quadric and (η_1, η_2, η_3) be any point of ϵ. If (p_1, p_2, p_3), (q_1, q_2, q_3) are two vectors parallel to ϵ, then the parametric equations of ϵ are

$$\xi_i = \eta_i + \zeta_1 p_i + \zeta_2 q_i, \qquad i = 1, 2, 3, \tag{19.37}$$

where ζ_1 and ζ_2 are parameters. Let the vectors (p_i) and (q_i) be chosen as unit vectors orthogonal to one another, so that

$$\sum p_i^2 = \sum q_i^2 = 1, \qquad \sum p_i q_i = 0$$

Therefore $$\zeta_1 = \sum p_i(\xi_i - \eta_i), \qquad \zeta_2 = \sum q_i(\xi_i - \eta_i)$$

This shows that $\zeta_1 = 0$ and $\zeta_2 = 0$ can be regarded as two planes passing through the point (η_i) and orthogonal to one another. Accordingly, we can regard the parameters ζ_1 and ζ_2 as the coordinates (ζ_1, ζ_2) of a point in the given plane ϵ with respect to the rectangular axes through the point (η_i), the positive axes of coordinates being in the directions of the vectors (p_i) and (q_i). Substituting (19.37) in (19.36), we get

$$\sum_{i,k=1}^{3} a_{ik}(\eta_i+\zeta_1 p_i+\zeta_2 q_i)(\eta_k+\zeta_1 p_k+\zeta_2 q_k)+2\sum_{i=1}^{3} a_i(\eta_i+\zeta_1 p_i+\zeta_2 q_i)+a = 0,$$

or,
$$\sum_{i,k=1}^{2} d_{ik}\zeta_i\zeta_k+2\sum_{i=1}^{2} d_i\zeta_i+d = 0, \text{ (say)} \qquad (19.38)$$

where d_{ik} depends on a_{ik}, p_i, q_i and $d_{ik} = d_{ki}$. The equation (19.38) therefore represents the curve of intersection of the given quadric (19.36) by the given plane (19.37). The section is a second degree curve, *i.e.*, a conic. The nature of this conic depends, as we have seen in § 12, on the coefficients d_{ik} and not on d_i or d, *i.e.*, it depends on a_{ik}, p_i and q_i.

We notice here that if it is desired to obtain a section by a plane parallel to ϵ, it is necessary to change only the quantities η_i in (19.37). Now since d_{ik} does not depend on η_i, it follows that *the sections of a quadric by parallel planes are conics of the same main type*, namely, either elliptic or hyperbolic or parabolic.

Let us return to (x, y, z) notation. We may recall from plane geometry that for the three main types, second degree equations can be transformed in the following forms :

(*i*) elliptic : $x^2/a^2+y^2/b^2-l(x, y) = 0$,
(*ii*) hyperbolic : $x^2/a^2-y^2/b^2-l(x, y) = 0$,
(*iii*) parabolic : $x^2-l(x, y) = 0$,

where the linear functions $l(x, y)$ may, by suitable transformation, be reduced to constants, positive, negative or zero. When the conic is non-degenerate, the ratio of its axes is independent of $l(x, y)$ in case (*i*), the angle between its asymptotes is independent of $l(x, y)$ in case (*ii*), and the direction of its axis is independent of $l(x, y)$ in case (*iii*). Hence, if parallel plane sections are ellipses, these ellipses have parallel axes and the same eccentricity. If parallel plane sections are hyperbolas, these hyperbolas have parallel asymptotes. And if parallel plane sections are parabolas, the axes of these parabolas are parallel.

Corollaries. (1) A plane section of a second degree cone is an ellipse, a parabola or a hyperbola according as a parallel plane through the vertex of the cone meets the cone in a point, in one line or in two distinct lines.

(2) A plane section of a hyperboloid is an ellipse, a parabola or a hyperbola according as it intersects the asymptotic cone of the hyperboloid in an ellipse, a parabola or a hyperbola.

(3) A plane section of a paraboloid $x^2/a^2 \pm y^2/b^2 = z$ is a parabola if and only if the plane is parallel to the z-axis ; otherwise the section is an ellipse or a hyperbola according as the paraboloid is elliptic or hyperbolic.

79. Circular section. I. *Circular sections of the hyperboloids*

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = \pm 1$$

By cor. (1), (2) of the last article it follows that a plane section of any one of the hyperboloids is a circle if the plane meets their asymptotic cone

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$$

in a circle which may be a point. Also if the section by a plane is a circle, all sections by parallel planes are circles. So we look for circular section of the asymptotic cone.

If $a = b$, the cone is a circular cone and therefore every section of either hyperboloid by a plane parallel to the (x, y)-plane is a circle.

If $a \neq b$, then $a \gtrless b$. For the sake of definiteness, let $a > b$. Consider the section of the cone by a plane ϵ defined parametrically by

$$\begin{aligned} x &= \eta_1 + \zeta_1 p_1 + \zeta_2 q_1 \\ y &= \eta_2 + \zeta_1 p_2 + \zeta_2 q_2 \\ z &= \eta_3 + \zeta_1 p_3 + \zeta_2 q_3 \end{aligned} \tag{19.39}$$

where the plane ϵ passes through a point (η_i) and is parallel to two orthogonal unit vectors (p_i), (q_i) and, as in the last article, the parameters ζ_1, ζ_2 are regarded as the coordinates of a point of ϵ. Substituting from (19.39) in the equation of the asymptotic cone, we obtain the equation of the plane section as

$$\sum_{i,k} d_{ik}\zeta_i\zeta_k + \sum_i d_i\zeta_i + d = 0, \qquad d_{ik} = d_{ki}, \qquad i,k = 1, 2, \tag{19.40}$$

where

$$\begin{aligned} d_{11} &= p_1^2/a^2 + p_2^2/b^2 - p_3^2/c^2, \\ d_{22} &= q_1^2/a^2 + q_2^2/b^2 - q_3^2/c^2, \\ d_{12} &= p_1q_1/a^2 + p_2q_2/b^2 - p_3q_3/c^2 \end{aligned}$$

The conditions that the plane section represents a circle are

$$d_{11} = d_{22}, \qquad d_{12} = 0$$

Now, for any plane ϵ, the unit vector (q_i) may be so chosen that we may, without loss of generality, suppose

$$q_3 = 0, \quad \text{so that} \quad q_1^2 + q_2^2 = 1$$

So put $$q_1 = \cos\phi \quad \text{and} \quad q_2 = \sin\phi$$

Since we are concerned with the ratio $p_1 : p_2 : p_3$, we may put

$$p_1 = t\sin\phi, \qquad p_2 = t\cos\phi$$

But as $$p_1^2+p_2^2+p_3^2 = 1, \qquad \text{therefore} \qquad t^2+p_3^2 = 1$$

So put $$t = \pm\cos\chi, \qquad p_3 = \sin\chi$$

Thus according to the above supposition, the coordinates of the orthogonal unit vectors (p_i) and (q_i) may be shown as follows :

$$\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ p_3 & q_3 \end{pmatrix} = \begin{pmatrix} \cos\chi\sin\phi & \cos\phi \\ -\cos\chi\cos\phi & \sin\phi \\ \sin\chi & 0 \end{pmatrix} \tag{19.41}$$

Hence the conditions for a circle can now be written as

$$\cos^2\chi(\sin^2\phi/a^2+\cos^2\phi/b^2)-\sin^2\chi/c^2 = \cos^2\phi/a^2+\sin^2\phi/b^2$$

and $$\cos\chi\,(\sin\phi\cos\phi/a^2-\sin\phi\cos\phi/b^2) = 0$$

These two conditions may finally be written as

$$\cos^2\chi(\sin^2\phi/a^2+\cos^2\phi/b^2+1/c^2) = \cos^2\phi/a^2+\sin^2\phi/b^2+1/c^2$$

and $$\cos\chi\sin 2\phi(1/a^2-1/b^2) = 0$$

The first of the conditions shows that $\cos\chi \neq 0$, and as $a \neq b$, it follows from the second that

$$\sin 2\phi = 0, \qquad \text{or} \qquad \phi = n\pi/2,$$

where n is an integer. Therefore two cases may arise :

(*i*) $\cos\phi = 0$, and so $\sin^2\phi = 1$

(*ii*) $\sin\phi = 0$, and so $\cos^2\phi = 1$

Taking case (*i*), we have, from the first of the conditions,

$$\cos^2\chi = (1/b^2+1/c^2)/(1/a^2+1/c^2)$$

Therefore, $\cos^2\chi>1$, because, by hypothesis, $a > b$. Hence this case must be rejected as $\cos^2\chi$ cannot be greater than 1.

Taking case (*ii*), we have

$$\cos^2\chi = (1/a^2+1/c^2)/(1/b^2+1/c^2)$$

This case is admissible. Therefore (19.41) can now be written as

$$\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ p_3 & q_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos\chi & 0 \\ \sin\chi & 0 \end{pmatrix} \text{ or } = \begin{pmatrix} 0 & -1 \\ \cos\chi & 0 \\ \sin\chi & 0 \end{pmatrix}, \tag{19.42}$$

according as $\cos\phi = +1$ or -1, where the values of $\cos\chi$ and $\sin\chi$ are to be obtained from the admissible case above. Hence there are two sets of values for (p_i) and (q_i) which determine the planes ϵ for circular sections of the asymptotic cone.

Thus *there are two systems of parallel planes which intersect the hyperboloids in circles.* Moreover, our supposition $a > b$ gives $(q_i) = (\pm 1, 0, 0)$; so, these two systems of planes are parallel to the x-axis.

Similarly, if we supposed $a < b$, only the case (i) would be admissible and the two systems of parallel planes which give circular sections would be parallel to the y-axis.

II. *Circular sections of*

$$\begin{array}{lll} \textit{ellipsoid}: & x^2/a^2+y^2/b^2+z^2/c^2 & = 1 \\ \textit{elliptic cylinder}: & x^2/a^2+y^2/b^2 & = 1 \\ \textit{elliptic paraboloid}: & x^2/a^2+y^2/b^2 & = z \end{array}$$

Let $a > b > c$. Consider the sections of the surfaces by the plane (19.39) where, as before, the quantities p_i, q_i are given by (19.41).

Substituting from (19.39) in the equations of the given surfaces, we obtain equations of the form (19.40) where it is now seen that

$$\left.\begin{array}{l} d_{12} = \tfrac{1}{2}\cos\chi \sin 2\phi(1/a^2 - 1/b^2) \\ d_{22} = \cos^2\phi/a^2 + \sin^2\phi/b^2 \end{array}\right], \quad \text{for all the three surfaces,}$$

$$d_{11} = \begin{cases} \cos^2\chi\ (\sin^2\phi/a^2 + \cos^2\phi/b^2) + \sin^2\chi/c^2, & \text{for the ellipsoid,} \\ \cos^2\chi\ (\sin^2\phi/a^2 + \cos^2\phi/b^2), & \text{for the elliptic cylinder and the elliptic paraboloid.} \end{cases}$$

The conditions for a circle in all cases are obviously $d_{11} = d_{22}$ and $d_{12} = 0$.

For a circular section of the ellipsoid, the first condition $d_{11} = d_{22}$ reduces to

$$\cos^2\chi\ (\sin^2\phi/a^2 + \cos^2\phi/b^2 - 1/c^2) = \cos^2\phi/a^2 + \sin^2\phi/b^2 - 1/c^2$$

Since $a > b > c$, the right hand side

$$\cos^2\phi/a^2 + \sin^2\phi/b^2 - 1/c^2 \leq (\cos^2\phi + \sin^2\phi)/b^2 - 1/c^2 < 0;$$

so the left hand side cannot vanish, and therefore $\cos\chi \neq 0$. And the second condition $d_{12} = 0$ reduces to

$$\cos\chi \sin 2\phi\ (1/a^2 - 1/b^2) = 0, \quad \text{or} \quad \sin 2\phi = 0$$

As before, there are two possibilities :

(i) $\sin\phi = 0$, and so $\cos^2\phi = 1$

(ii) $\cos\phi = 0$, and so $\sin^2\phi = 1$

Taking case (*i*), we have, from the first of the conditions,

$$\cos^2\chi = (1/a^2 - 1/c^2)/(1/b^2 - 1/c^2) > 1$$

This case is therefore inadmissible. Taking case (*ii*), we get

$$\cos^2\chi = (1/b^2 - 1/c^2)/(1/a^2 - 1/c^2)$$

This case is admissible. Hence we obtain circular sections of the ellipsoid when the values of (p_i) and (q_i) are given by

$$\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ p_3 & q_3 \end{pmatrix} = \begin{pmatrix} \cos\chi & 0 \\ 0 & 1 \\ \sin\chi & 0 \end{pmatrix} \text{ or } = \begin{pmatrix} -\cos\chi & 0 \\ 0 & -1 \\ \sin\chi & 0 \end{pmatrix}, \tag{19.43}$$

the values of $\cos\chi$ and $\sin\chi$ being given by the admissible case above.

Thus, *there are two systems of parallel planes which give circular sections of an ellipsoid.* For the given equation of the ellipsoid, the two systems of planes are parallel to the y-axis.

For circular sections of the elliptic cylinder and the elliptic paraboloid, the conditions are

$$\cos^2\chi(\sin^2\phi/a^2 + \cos^2\phi/b^2) = \cos^2\phi/a^2 + \sin^2\phi/b^2$$

and
$$\cos\chi \sin 2\phi(1/a^2 - 1/b^2) = 0$$

As before
$$\cos\chi \neq 0, \quad \text{and so} \quad \sin 2\phi = 0$$

The two possibilities are :

(*i*) $\cos\phi = 0, \quad \text{and so} \quad \sin^2\phi = 1$

Accordingly $\cos^2\chi = a^2/b^2 > 1$. This case is therefore inadmissible.

(*ii*) $\sin\phi = 0, \quad \text{and so} \quad \cos^2\phi = 1$

Accordingly $\cos^2\chi = b^2/a^2$. This case is therefore admissible.
Hence we obtain circular sections of the elliptic cylinder and the elliptic paraboloid when (p_i) and (q_i) have the values

$$\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ p_3 & q_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b/a & 0 \\ \pm\sqrt{1 - b^2/a^2} & 0 \end{pmatrix} \tag{19.44}$$

Thus, *there are two systems of parallel planes which give circular sections of an elliptic cylinder and of an elliptic paraboloid.*

We may obtain the equations of the planes which give the circular sections in the following way :

Eliminate the parameters ζ_1 and ζ_2 between the equations (19.39) and obtain the equation of the plane ϵ as

$$(p_2q_3-p_3q_2)x+(p_3q_1-p_1q_3)y+(p_1q_2-p_2q_1)z = \text{a constant.}$$

Now take, for instance, the case of the ellipsoid ; the cases of the other two surfaces are to be dealt with similarly. From (19.43) and the above equation of the plane, the circular sections are given by the planes

$$\pm x \sin \chi + z \cos \chi = \text{a constant,}$$

where $$\cos^2 \chi = a^2(b^2-c^2)/b^2(a^2-c^2)$$

Hence the two systems of parallel planes which give circular sections of the ellipsoid are

$$\frac{x}{a}\sqrt{a^2-b^2} \pm \frac{z}{c}\sqrt{b^2-c^2} = \text{suitable arbitrary constants.} \qquad (19.45)$$

As an *application* of these circular sections, put

$$S \equiv x^2/a^2+y^2/b^2+z^2/c^2-1,$$

$$u \equiv \frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}-\mu, \qquad v \equiv \frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2}-\nu,$$

where μ and ν are two constants. Then the equation

$$\gamma S+\lambda uv = 0,$$

where γ, λ are arbitrary constants, is satisfied by

$$S=0=u \quad \text{and} \quad S=0=v$$

So the equation represents a quadric passing through the two non-parallel circular sections of the ellipsoid. Putting the particular values $\gamma = b^2$, $\lambda = 1$, the equation reduces to the form

$$x^2+y^2+z^2+Ax+Bz+C = 0$$

This shows that *two non-parallel circular sections of an ellipsoid always lie on a sphere.*

Any line (plane) which passes through the centre of an ellipsoid is called a *diameter* (a *diametral plane*) of the ellipsoid. Analogous to the properties of diameters of an ellipse in the plane geometry, we have the following properties :

(1) The centres of parallel sections of an ellipsoid lie on a diameter.

(2) The middle points of a set of parallel chords of an ellipsoid lie on a diametral plane.

(3) If a diameter d of an ellipsoid contains the centres of sections parallel to a diametral plane ϵ, then ϵ bisects the chords parallel to d, and conversely. The relationship between d and ϵ is said to be *conjugate*. The tangent planes at the extremities of d are parallel to ϵ.

An *umbilic* of a quadric which has circular sections is an extremity of a diameter which contains the centres of circular sections. There are therefore four umbilics of an ellipsoid. Let (x_0, y_0, z_0) be the coordinates of an umbilic of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

Equation of the tangent plane to the surface at (x_0, y_0, z_0) is

$$x_0x/a^2 + y_0y/b^2 + z_0z/c^2 = 1$$

Hence, since the coefficients of x, y, z of this equation and equation (19.45) must be proportional, we have

$$x_0/a = k\sqrt{a^2-b^2}, \quad z_0/c = \pm k\sqrt{b^2-c^2}, \quad y_0 = 0,$$

where $k \neq 0$ is constant. But as (x_0, y_0, z_0) lies on the ellipsoid, $k^2 = 1/(a^2-c^2)$. Therefore the coordinates of the four umbilics of the ellipsoid are given by

$$x_0 = \pm a\sqrt{a^2-b^2}/\sqrt{a^2-c^2}, \quad y_0 = 0, \quad z_0 = \pm c\sqrt{b^2-c^2}/\sqrt{a^2-c^2} \quad (19.46)$$

As an *application* of (19.46) it is easily seen that these four umbilics lie on the sphere

$$x^2+y^2+z^2 = a^2-b^2+c^2.$$

80. Confocal quadrics. Consider all quadrics of the types

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1, \qquad (19.47)$$

where a^2, b^2, c^2 are unequal positive constants and λ is a parameter. Without loss of generality, we may suppose $a^2 > b^2 > c^2$. If $c^2 > \lambda$, the surfaces (19.47) are all ellipsoids. Since the sections of these ellipsoids by each one of their principal planes, *i.e.*, the planes $z = 0, y = 0, x = 0$, are confocal ellipses, the surfaces (19.47) of the type given by the prescribed values of λ are confocal ellipsoids. If $b^2 > \lambda > c^2$, all surfaces (19.47) of this type are confocal hyperboloids of one sheet. And if $a^2 > \lambda > b^2$, all surfaces (19.47) form a type of confocal hyperboloids of two sheets.

Further, all the three types of surfaces have the common property that their sections by the plane $z = 0$ are confocal conics, the common foci being at the points

$$(\pm\sqrt{a^2 - b^2}, \quad 0, \quad 0)$$

Similarly, for sections of all the three types of surfaces by the plane $y = 0$ and for sections of the first two types by the plane $x = 0$. On account of these properties, the three types of surfaces are called *confocal quadrics*. Evidently, these confocal quadrics exist unless λ takes one of the values a^2, b^2, c^2 or $\lambda > a^2$.

Let $P = (x_0, y_0, z_0)$ be a point other than a point of the coordinate planes. Those of the three types of surfaces (19.47) which pass through P satisfy the equation

$$x_0^2/(a^2 - \lambda) + y_0^2/(b^2 - \lambda) + z_0^2/(c^2 - \lambda) = 1$$

This is a cubic equation in λ and so has three roots, say $\lambda_1, \lambda_2, \lambda_3$. In order that all the roots be real, it can be seen from algebraical consideration that

$$a^2 > \lambda_1 > b^2 > \lambda_2 > c^2 > \lambda_3,$$

when $\lambda_1 > \lambda_2 > \lambda_3$. But this is exactly the condition that gives three surfaces, one of each type.

Hence, *through each point of the space, with the exception of those of the principal planes, there pass three confocal quadrics, one of each type.*

Take any two of these confocal quadrics, say those corresponding to λ_1 and λ_2. The tangent planes to them at their common point P are given by the equations

$$x\,x_0/(a^2 - \lambda_1) + y\,y_0/(b^2 - \lambda_1) + z\,z_0/(c^2 - \lambda_1) = 1,$$

$$x\,x_0/(a^2 - \lambda_2) + y\,y_0/(b^2 - \lambda_2) + z\,z_0/(c^2 - \lambda_2) = 1$$

And since the surfaces pass through P, we have

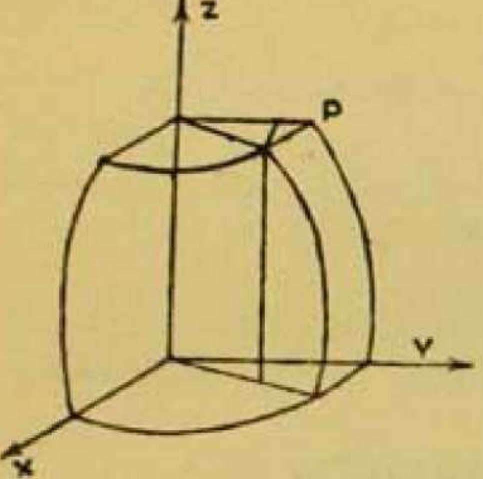
$$x_0^2/(a^2 - \lambda_1)\,(a^2 - \lambda_2) + y_0^2/(b^2 - \lambda_1)\,(b^2 - \lambda_2) + z_0^2/(c^2 - \lambda_1)\,(c^2 - \lambda_2) = 0$$

But this is the condition that the two tangent planes be orthogonal.

Thus, *the three confocal quadrics intersect orthogonally all along their curve of intersection. Therefore the confocal quadrics form a triply orthogonal system of surfaces.*

81. Surfaces of revolution and ruled surfaces. *A surface of revolution is a surface which is generated by a plane curve rotating about a straight line lying in the plane of the curve.*

Take the axis of rotation as the z-axis and the generating curve to lie originally in the plane $x = 0$. For simplicity, let the equation of the curve be $y^2 = f(z)$, a function of z, so that the curve is symmetrical about the z-axis.



Let a point $P = (0, y', z)$ on the curve be moved to the point (x, y, z) after rotation about the z-axis. Then

$$y'^2 = x^2 + y^2$$

The equation of the surface of revolution is therefore

$$x^2 + y^2 = f(z)$$

For example, the ellipse $y^2/b^2 + z^2/c^2 = 1$ in the plane $x = 0$, by rotation about the z-axis, generates the surface of revolution

$$(x^2 + y^2)/b^2 + z^2/c^2 = 1,$$

which is called an *ellipsoid of revolution* or a *spheroid*. It is an *oblate* spheroid if $b < c$ and a *prolate* spheroid if $b > c$. It is, in particular, a sphere if $b = c$; so a circle, by rotation about one of its diameters, generates a sphere. Again, the hyperbola $-y^2/b^2 + z^2/c^2 = 1$ in the plane $x = 0$ generates, by revolution about the z-axis, the surface

$$-(x^2 + y^2)/b^2 + z^2/c^2 = 1,$$

which is known as a *hyperboloid of revolution of two sheets* and consists of two different parts. The hyperbola $y^2/b^2 - z^2/c^2 = 1$ in the plane $x = 0$ generates in the same way the surface

$$(x^2 + y^2)/b^2 - z^2/c^2 = 1,$$

which is a *hyperboloid of revolution of one sheet*. Finally, the parabola $y^2 = 4\,pz$ in the plane $x = 0$, by revolution about the z-axis, generates the *paraboloid of revolution* $x^2 + y^2 = 4\,pz$.

A straight line rotated about an axis in the same plane with the line will generate a surface of the second degree. If the equation of the line is, for example, $y = \lambda z$ in the plane $x = 0$, the surface generated by revolution about the z-axis is the *circular cone* $x^2 + y^2 = \lambda^2 z^2$. Similarly, the line $y = b$ generates the *circular cylinder* $x^2 + y^2 = b^2$.

A surface may be generated by the motion of a line or a plane. In the former case, we get a *ruled surface*, and in the latter *an envelope*. It is evident that cones and cylinders are ruled surfaces. Let us see how a hyperboloid of one sheet can be generated by the motion of a line.

Consider two equal and similarly placed ellipses on two parallel planes and let the line joining their centres be perpendicular to the planes. Let this perpendicular be taken as the axis of z, the middle point of the segment joining the centres as the origin and the axes of x and y be parallel to the axes of the ellipses. Let the equations of the ellipses be

$$x^2/a^2 + y^2/b^2 = 1, \qquad z = c$$

and

$$x^2/a^2 + y^2/b^2 = 1, \qquad z = -c$$

Take points Q and Q' on the ellipses such that their eccentric angles (§ 24.2) are $\phi - \psi$ and $\phi + \psi$ respectively and let the line QQ' move in such a manner that the difference of these eccentric angles remains constant, so that ψ is constant. Then the coordinates of Q and Q' are respectively

$$[a \cos(\phi - \psi), \quad b \sin(\phi - \psi), \quad c]$$

and

$$[a \cos(\phi + \psi), \quad b \sin(\phi + \psi), \quad -c]$$

Therefore

$$\overline{QQ'} = (-2a \sin\phi \sin\psi, \quad 2b \cos\phi \sin\psi, \ -2c)$$

If $P = (x, y, z)$ is any point on the line QQ', then

$$x = a \cos(\phi - \psi) + \rho\, a \sin\phi \sin\psi$$

$$y = b \sin(\phi - \psi) - \rho\, b \cos\phi \sin\psi$$

$$z = c\,(1 + \rho)$$

Therefore

$$x^2/a^2 + y^2/b^2 = \cos^2\psi + (1 + \rho)^2 \sin^2\psi$$

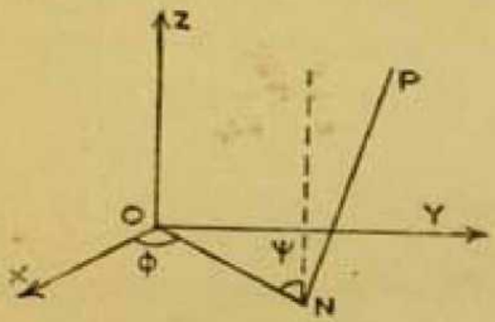
Hence, eliminating the parameters ϕ and ρ, we have the locus of P as

$$x^2/a^2 + y^2/b^2 - z^2/c^2 \operatorname{cosec}^2\psi = \cos^2\psi$$

As ψ is constant, the locus is a hyperboloid of one sheet. Thus *the surface generated by a line which joins pairs of points having constant difference of eccentric angles on two equal and similarly placed ellipses in parallel planes is a hyperboloid of one sheet.* If the sign of ψ is changed, the equation of the surface is unaltered ; hence the surface is covered by two reguli.

A hyperboloid of revolution of one sheet can be generated by a straight line which rotates about an axis such that the generating line and the axis are non-coplanar. This can be seen as follows :

Let ON be the common perpendicular of the axis of rotation and the generating line in any position meeting them in O and N respectively ; then $|ON| = p$, a constant. Take the axis of rotation as the z-axis, O as the origin and any two fixed perpendicular lines in the plane through O normal to the z-axis as the axes of x and y. Let the angle between the axis of rotation and the generating line be ψ ; then ψ is constant. The coordinates of N can then be taken as



$$(p \cos \phi, \quad p \sin \phi, 0)$$

If $P = (x, y, z)$ is any point on the generating line and $|NP| = \rho$, then

$$x = p \cos \phi - \rho \sin \psi \sin \phi$$

$$y = p \sin \phi + \rho \sin \psi \cos \phi$$

$$z = \rho \cos \psi$$

where ϕ and ρ are parameters. Eliminating these parameters, we get the locus of P as the surface

$$x^2 + y^2 - z^2 \tan^2\psi = p^2$$

As p and ψ are constants, the surface is a hyperboloid of revolution of one sheet.

A hyperbolic paraboloid can be generated by the motion of a line in the following way : Let $AB\,A'B'$ be a regular tetrahedron, and Q and Q' be two variable points on two opposite edges AB and $A'B'$ such that

$$|AQ| = |A'Q'|$$

Then the line QQ' generates a hyperbolic paraboloid. This can be seen as follows :

Let the length of half the edges of the regular tetrahedron be $d\sqrt{2}$ and $|AQ| = |A'Q'| = r\sqrt{2}$. Since the tetrahedron is regular, the three lines joining the middle points of the three pairs of opposite sides intersect orthogonally in a point O, say, which is equidistant, k say, from these middle points. Choose these three lines as the coordinate axes, O being the origin. Then the coordinates of Q and Q' can be taken respectively as

$$(d-r,\ k,\ d-r) \text{ and } (d-r,\ -k,\ -d+r)$$

Therefore, if $P=(x, y, z)$ is any point of the line QQ', we get

$$x=d-r, \quad y=k(1+\rho), \quad z=(d-r)(1+\rho)$$

Eliminating the parameters d, r, ρ, we get $xy=kz$. This equation represents a hyperbolic paraboloid ; for, by a suitable orthogonal transformation, the equation (19.20) can be transformed into this form. It can be seen that the equation of the surface remains unaltered if Q, Q' satisfy $|AQ|=|B'Q'|$; this shows that the surface has two system of generators.

CHAPTER XX

LAW OF INERTIA FOR QUADRATIC FORMS

82. Homogeneous quadratic form. The general homogeneous quadratic form in n variables $x_1, \ldots, x_n$ is

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j, \qquad a_{ij} = a_{ji} \tag{20.1}$$

The condition that the matrix (a_{ij}) is symmetric is a matter of convenience involving no loss of generality. As we shall be concerned with real quadratic forms, the coefficients a_{ij} are supposed to be real numbers and at least one of them is supposed to be different from zero. Let

$$x'_i = \sum_{j=1}^{n} t_{ij} x_j, \qquad i = 1, \ldots . n \tag{20.2}$$

be a real linear transformation of the variables. The transformation is nonsingular if the rank of the matrix (t_{ij}) is n, *i.e.*, $| t_{ij} | \neq 0$, otherwise it is singular. We first establish the following theorem which is a generalisation of the theorem given in § 61.

Theorem 1. Any given real quadratic form (20.1) can be transformed by a nonsingular linear transformation into a normal form

$$x_1^2 + \ldots + x^2_p - x^2_{p+1} - \ldots - x_r^2, \qquad r \leq n \tag{20.3}$$

Proof: If the matrix (a_{ij}) has a diagonal term $a_{kk} \neq 0$, we apply the transformation defined by

$$x'_1 = x_k, \quad x'_k = x_1, \quad x'_m = x_m, \quad m \neq 1, k$$

This transformation transforms the form (20.1) into, say,

$$a'_{ij} x'_i x'_j \qquad \text{where} \qquad a'_{11} = a_{kk} \neq 0$$

If all the diagonal terms of (a_{ij}) are zero, but there is a term $a_{kl} \neq 0$, $k \neq l$, we take the transformation $T = T_2 T_1$ given as a product of two others defined by

$$T_1: \qquad x'_1 = x_k, \quad x'_2 = x_l, \quad x'_m = x_t, \qquad m \neq 1, 2\,; t \neq k, l\,;$$

$$\text{and } m = t \text{ only when } m \neq 1, 2, k,$$

$$T_2: \qquad x''_1 = \tfrac{1}{2}(x'_1 + x'_2), \quad x''_2 = \tfrac{1}{2}(x'_1 - x'_2), \qquad x''_s = x'_s, \quad s \neq 1, 2$$

Then (20.1) is transformed into, say,

$$\Sigma a''_{ij} x''_i x''_j, \qquad \text{where} \qquad a''_{11} = 2a_{kl} \neq 0$$

It is thus seen that we can, without loss of generality, suppose that the leading coefficient a_{11} of (20.1) is different from zero and therefore (20.1) can be written as

$$a_{11}\left(\sum_{i,j=1}^{n} b_{ij} x_i x_j\right), \qquad \text{where} \qquad b_{ij} = a_{ij}/a_{11} \qquad (20.4)$$

The terms involving x_1 in $\Sigma\, b_{ij} x_i x_j$ are

$$x_1^2 + 2x_1 \sum_{j=1}^{n} b_{1j} x_j = \left(x_1 + \sum_{j=2}^{n} b_{1j} x_j\right)^2 - \left(\sum_{j=2}^{n} b_{1j} x_j\right)^2$$

Now apply the transformation

$$x'_1 = x_1 + \sum_{j=2}^{n} b_{1j} x_j \qquad x'_j = x_j, \;\; j = 2, \ldots, n$$

Then (20.4) is transformed into the form

$$a_{11} x_1'^2 + \sum_{j,k=2}^{n} c_{jk} x'_j x'_k$$

If the residual part $\Sigma c_{jk} x'_j x'_k$ is not identically zero, it can be treated in a similar manner. Repeating this process, the quadratic form (20.1) will, after a finite number of steps, be transformed into (dropping the dashes)

$$d_1 x_1^2 + \ldots + d_r x_r^2, \quad \text{where} \quad d_i \neq 0 \quad i = 1, \ldots, r\,; \quad r \leq n \qquad (20.5)$$

Ultimately, applying the transformation

$$x'_i = \sqrt{|d_i|}\, x_i, \qquad x'_j = x_j \qquad i = 1, \ldots, r\,; \qquad j = r+1, \ldots, n$$

and the permuting, if necessary, the variables so that the positive terms come first, the form (20.5) is transformed into the required form (20.3). All the transformations used above are nonsingular linear transformations and so is their product. Hence the theorem.

Secondly, we have the following theorem :

Theorem 2. The number r of terms in the normal form (20.3) is equal to the rank of the matrix (a_{ij}) of the coefficients of the form (20.1).

Proof : It is sufficient to show that the rank of (a_{ij}) remains invariant by any nonsingular linear tranformation. If (20.2) is nonsingular, its inverse exists which is also nonsingular. Let the inverse be

$$x_i = \sum_j t'_{ij} x_j, \qquad i = 1, \ldots, n$$

This transformation transforms (20.1) into, say,

$$\sum_{i,j} a'_{ij} x'_i x'_j \qquad \text{where} \qquad a'_{ij} = \sum_{k,l} t'_{ki} a_{kl} t'_{lj}$$

It can therefore be seen that

$$(a'_{ij}) = (t'_{ij})^T (a_{ij}) (t'_{ij}),$$

where $(t'_{ij})^T$ is the transposed of (t'_{ij}). It now follows from theorem 23, Chapter 0, that the matrices (a_{ij}) and (a'_{ij}) have the same rank. Hence the theorem.

83. Law of inertia. Finally, we give the following theorem which is known as *Sylvester's law of inertia.*

Theorem 3. The number p of positive terms in the normal form (20.3) is an invariant of the form (20.1) under nonsingular linear transformations.

Proof : Let the form (20.1) be transformed into the forms

$$\begin{aligned} & x'^2_1 + \ldots + x'^2_p - x'^2_{p+1} - \ldots - x'^2_r \\ \text{and} \qquad & x''^2_1 + \ldots + x''^2_q - x''^2_{q+1} - \ldots - x''^2_r \end{aligned} \tag{20.6}$$

by the nonsingular linear transformations

$$\begin{aligned} & x'_i = l'_i(x_1, \ldots, x_n) \\ \text{and} \qquad & x''_i = l''_i(x_1, \ldots, x_n) \end{aligned} \qquad i = 1, \ldots, n \tag{20.7}$$

respectively, where the l'_i's and l''_i's are homogeneous linear functions in the variables $x_1, \ldots, x_n$. Obviously, if we substitute l'_i for x'_i in the first of the forms (20.6), we get the form (20.1) by virtue (20.7) ; similarly the second of the forms (20.6) reduces to (20.1) by the substitution l''_i for x''_i. Therefore we may regard both the forms (20.6) as identically equal to the form (20.1) and hence to each other. So we may put

$$\begin{aligned} & x'^2_1 + \ldots + x'^2_p - x'^2_{p+1} - \ldots - x'^2_r \\ & = x''^2_1 + \ldots + x''^2_q - x''^2_{q+1} - \ldots - x''^2_r \end{aligned} \tag{20.8}$$

If possible, let p and q be unequal. We shall show that this leads to contradiction. Let us then suppose, without loss of generality, that

$p > q$. Consider the following system of $n-(p-q)$ linear homogeneous equations in $x_1, \ldots, x_n$:

$$\begin{aligned} &l'_{p+1}(x_1, \ldots, x_n) = 0, \ldots, l'_n(x_1, \ldots, x_n) = 0, \\ &l''_1(x_1, \ldots, x_n) = 0, \ldots, l''_q(x_1, \ldots, x_n) = 0 \end{aligned} \qquad (20.9)$$

As the number of equations is less than n, there exists a solution, say $(c_1, \ldots, c_n)$, not all zero, of the system (20.9). Let

$$\begin{aligned} &l'_1(c_1, \ldots, c_n) = c'_1, \ldots, l'_p(c_1, \ldots, c_n) = c'_p, \\ &l''_{q+1}(c_1, \ldots, c_n) = c''_{q+1}, \ldots, l''_r(c_1, \ldots, c_n) = c''_r, \end{aligned}$$

where the c''s and c'''s are obviously real numbers. As $(c_1, \ldots, c_n)$ is a solution of (20.9), the equation (20.8) now becomes

$$c'^2_1 + \ldots + c'^2_p = -c''^2_{q+1} - \ldots - c''^2_r$$

Further as the expression on the left-hand side cannot be negative and that on the right cannot be positive, both the expressions must be zero.

So $$c'_1 = 0, \ldots, c'_p = 0$$

These together with the first line of (20.9) give

$$l'_1(c_1, \ldots, c_n) = 0, \ldots, l'_n(c_1, \ldots, c_n) = 0$$

This shows that $(c_1, \ldots, c_n)$ is a solution of the system of n linear homogeneous equations

$$l'_i(x_1, \ldots, x_n) = 0, \qquad i = 1, \ldots, n$$

As $c_1, \ldots, c_n$ are not all zero, the determinant of the coefficients of this system must be zero. This implies that the first of the linear transformations (20.7) is singular which is contrary to hypothesis. Hence the theorem.

We can thus associate with every real quadratic form (20.1) the two integral invariants r and p and therefore also the invariant $r-p$ which is the number of negative terms in the normal form (20.3). It is however found more convenient to use the invariant $s = 2p - r$ which is the difference of the number of positive and the negative terms and called the *signature* of the quadratic form (20.1). In particular, the quadratic form is said to be *positive definite* when $s = r = n$.

84. Geometrical significance. As applications to geometry, it is seen, as in § 57, that a nonsingular linear transformation (20.2) represents a collineation of an $(n-1)$-dimensional projective space when homogeneous coordinates are used and represents an affinity, with the origin remaining fixed, of an n-dimensional space when nonhomogeneous coordinates are used. In the projective space, a quadratic form (20.1) equated to zero represents a *hyperquadric*, and the normal form (20.3) equated to zero furnishes the projective classification of hyperquadrics, as in § 61. It may be stated, without proof, that using nonhomogeneous coordinates and orthogonal transformations, the classification of hyperquadrics in an n-dimensional metric space is given by

$$\begin{aligned} &(i) \quad a_1x_1^2 + \dots + a_nx_n^2 + 1 = 0, \\ &(ii) \quad a_1x_1^2 + \dots + a_{n-1}x_{n-1}^2 + x_n = 0, \qquad (20.10) \\ &(iii) \quad a_1x_1^2 + \dots + a_nx_n^2 = 0, \end{aligned}$$

where the a's are real numbers which are uniquely determined in (i), have an arbitrary factor ± 1 in (ii) and are determined, except for an arbitrary factor, in (iii).

EXAMPLES

The numbers O, I, II, III etc. refer to chapters.

O

1. Given m n-vectors. Show that if there exist among them $r < m$ dependent vectors, then the m vectors are dependent; and if the m vectors are independent, then any $r < m$ of them are also independent.

2. Show that the vector space generated by the 3-vectors $(1, 5, -3)$, $(2, 1, 1)$, $(-1, 4, 1)$ and that generated by $(1, -4, 4)$, $(2, 1, 1)$, $(2, 10, -1)$ are the same. What is the rank of this vector space?

3. Prove that the rank of the following matrix is either 0 or 3:

$$\begin{pmatrix} 0 & 0 & 0 & b & c & d \\ 0 & -d & c & -a & 0 & 0 \\ d & 0 & -b & 0 & -a & 0 \\ -c & b & 0 & 0 & 0 & -a \end{pmatrix}$$

4. Solve, if possible, the following equations:

$$\text{(i)}\quad \begin{aligned} 3x_1 - 17x_2 + x_3 &= 9 \\ 5x_1 + 5x_2 + 3x_3 &= 7 \\ 2x_1 - 3x_2 + x_3 &= 4 \end{aligned} \qquad \text{(ii)}\quad \begin{aligned} 5x_1 + 4x_2 - 8x_3 &= 9 \\ 2x_1 - 3x_2 + 6x_3 &= 2 \\ 7x_1 + 2x_2 - 4x_3 &= 1 \end{aligned}$$

5. If A_i, B_i, C_i are respectively the cofactors of a_i, b_i, c_i, $i=1, 2, 3$, in

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \text{prove that} \quad \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = D^2. \text{ Generalise.}$$

6. (*i*) Examine whether the following permutations are even or odd:

$$\begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}, \quad \begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix}, \quad \begin{pmatrix} a & b & c & d \\ d & c & a & b \end{pmatrix}, \quad \begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}$$

(*ii*) If $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 6 & 7 & 2 & 1 & 4 \end{pmatrix}$, show that P^6 is the identity.

(*iii*) If $P = (1, 3, 5, 6, 2)$ and $Q = (1, 4, 6)$ are cyclic permutations of six objects, find

$$PQ, \quad QP, \quad P^{-1}QP, \quad Q^{-1}PQ$$

I

1. If C is a point of a straight line AB such that $r\overline{AC} = s\overline{CB}$, where r and s are numbers, and O is any arbitrary point, show that $r\overline{OA}+s\overline{OB} = (r+s)\overline{OC}$.

2. If G is the centroid of the triangle ABC and O is any point in the plane, show that $3\overline{OG} = \overline{OA}+\overline{OB}+\overline{OC}$.

3. Deduce the trigonometrical formula $a = b \cos C+c \cos B$, from the vector equation $\overline{BC} = \overline{BA}+\overline{AC}$.

4. BC is perpendicular to AB in the positive direction, and CD is perpendicular to BC in the negative direction. If $AB : BC : CD = r : s : t$ and A, B have the coordinates $(0, 0)$, (x_1, y_1), find the coordinates of D.

5. Find the Hessian normal forms of the following equations of straight lines : (i) $3x+4y = 7$, (ii) $-2x+7y-5 = 0$, (iii) $3x+8 = 0$, (iv) $3x-7y = 0$. Find the areas of the triangles formed by the straight lines (i), (ii) and (iv) and the straight lines (i), (iii) and (iv).

6. Find the coordinates of the orthocentre of the triangle formed by the lines

$$y = m_i x+a/m_i, \qquad i = 1, 2, 3$$

7. Show that the two straight lines joining the points (1, 1), (2, 2) respectively to the point of intersection of the straight lines

$$19x+3y-29 = 0 \quad \text{and} \quad 13x+11y-27 = 0$$

are at right angles and find their internal and external bisectors.

8. Show that, as λ takes all values, the straight lines

$$(1+2\lambda)x-(1-3\lambda)y+2-\lambda = 0$$

form a pencil. Find the parametric equations of the line of the pencil which is parallel to the vector $(1, -1)$.

II

1. Calculate the cross-ratio (AB, CD), when it is possible, where

   (i) $A = (2, 3)$, $B = (4, 9)$, $C = (3, 6)$, $D = (10/3, 7)$;

   (ii) $A = (-3, 7)$, $B = (2, -5)$, $C = (9/7, -23/7)$, $D = (0, -1/5)$;

   (iii) $A = (5, 2)$, $B = (-13, 8)$, $C = (3, 7)$, $D = (4, -11)$.

2. Calculate the cross-ratio (ab, cd), where the equations of the lines a, b, c, d are respectively given by

   (i) $2x-3y = 0$, $5x+y = 0$, $x = 0$, $x+11y = 0$;

   (ii) $x+y-7 = 0$, $3x-2y+4 = 0$, $x = 2$, $4x-y-3 = 0$.

3. Given $(ab, cd) = -3/7$, and that the equations of

   (i) $a : x+y = 0$, $b : 2x-5y = 0$, $c : y = 0$, find the equation of d;

   (ii) $a : x+y = 0$, $b : 2x-5y = 0$, $d : y = 0$, find the equation of c;

   (iii) $a : x+y = 0$, $c : 2x-5y = 0$, $d : y = 0$, find the equation of b;

   (iv) $b : x+y = 0$, $c : 2x-5y = 0$, $d : y = 0$, find the equation of a.

4. Given two collinear segments AB and $A'B'$, determine another segment CD which shall divide each of them harmonically.

5. Given three collinear points A, B, C, find by geometrical construction a fourth point D such that (AB, CD) is equal to a given rational number r. Can the construction be simplified when $r = 1$?

6. If θ be the angle of intersection of the two circles described on the collinear segments AB and CD as diameters (it being supposed that A and B are separated by C and D), show that $(AB, CD) = -\tan^2(\theta/2)$.

7. If $(AB, CD) = (AB', C'D')$ on two different lines AD, AD', and if BB', CC' meet in O, prove that DD' must pass through O.

8. AA', BB', CC' are concurrent straight lines through the vertices of a triangle ABC meeting the opposite sides in A', B', C'. If $B'C'$ meets BC in A'', $C'A'$ meets CA in B'', $A'B'$ meets AB in C'', prove that $(BC, A'A'') = (CA, B'B'') = (AB, C'C'') = -1$.

9. Prove that $(AB, CD)(AB, DE) = (AB, CE)$, where A, B, C, D, E are five distinct points of a straight line. Hence deduce that

$$(AB, CD)(AB, DE)(AB, EC) = 1.$$

10. If A, B, C, D are four collinear points, show that

$$(BC, AD)(CA, BD)(AB, CD) = -1.$$

11. If $(AB, CD) = -1/3$ and B is the point of trisection of AD towards D, show that C is the other point of trisection of AD.

12. Prove that the cross-ratio of the pencil joining the fixed points $(at_i^2, 2at_i)$, $i = 1, 2, 3, 4$, to the variable point $(at^2, 2at)$ is independent of t.

13. A, B, C, D and A', B', C', D' are two tetrads of points on two straight lines and O, O' are points on AA'. If the points of intersection of OB, $O'B'$; OC, $O'C'$; OD, $O'D'$ are collinear, show that $(AB, CD) = (A'B', C'D')$.

III

1. Find the equations of the rigid motion which introduces the point $(1, 1)$ as the new origin and the straight line $2x+3y-5 = 0$ as the new x'-axis.

2. T is the transformation

$$x' = (\sqrt{3}/2)x-(1/2)y$$
$$y' = (1/2)x+(\sqrt{3}/2)y$$

Find T^3 and T^6 and give geometrical interpretations.

3. Given two perpendicular straight lines through the point (c_1, c_2) of slopes $1/2$ and -2. Find the rigid motion transforming these lines as axes.

4. If OP and OQ are any two lines and T is any translation, prove that there exists a unique pair of translations T', T'', parallel to OP and OQ respectively such that $T'T'' = T$.

5. Find suitable translations T, T' and rotations R, R' about the origin such that $RT = T'R' = M$, where M is given by

$$x' = (3/5)x-(4/5)y+3$$
$$y' = (4/5)x+(3/5)y+1.$$

6. Show that

$$\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$$

is an invariant of the point (x_1, y_1) and the straight line $ax+by+c=0$ with respect to any rigid motion. What is the geometrical significance ?

7. Show that

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

is an invariant of the points (x_1, y_1), (x_2, y_2) with respect to any rotation about the origin.

8. Show that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

is an invariant of the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) with respect to any rigid motion. What is the geometrical significance of this invariance ?

9. *ABC* and *DAC* are two triangles, *B* and *D* being on the opposite sides of *AC* and $|AB| = |AD|$. Find a transformation by which the triangle *ABC* is transformed into the triangle *DBC*. Does there exist a rigid motion giving the same result ?

IV

1. Reduce the following equation to the normal form by applying rigid motions

$$5x^2-2xy+5y^2-8x-8y-8=0\,;$$

and find the equations of the single rigid motion by which the given equation is reduced to the normal form by a single step.

2. Prove that any two lines drawn from a point outside a conic which are conjugate with respect to the conic are harmonically separated by the tangents from the same point to the conic. Deduce that any two points which are conjugate with respect to a conic and which lie on a line intersecting the conic in two points are harmonically separated by the points of intersection

3. Prove that the tangents to a central conic at the extremities of a chord meet on the diameter bisecting the chord. Is the proposition true for a parabola ?

V

1. Let g_1 and g_2 be two intersecting straight lines and T_1 and T_2 be orthogonal reflexions in g_1 and g_2 respectively. Compare the products T_1T_2 and T_2T_1. What are these product transformations when, in particular, g_1 and g_2 are perpendicular ?

2. Prove that the lines through the vertices of a triangle parallel to the transforms of the opposite sides by a given orthogonal line reflexion are concurrent.

3. Prove that a translation can be expressed as a product of two orthogonal reflexions in parallel straight lines.

4. Prove that a rotation about the origin through an angle θ is the resultant of two orthogonal reflexions in lines through the origin so chosen that the angle from the first to the second is $\theta/2$.

5. Describe the nature of the transformations :

$$\text{(i)}\quad \begin{aligned} x' &= -y+5 \\ y' &= x+1 \end{aligned} \qquad \text{and} \qquad \text{(ii)}\quad \begin{aligned} x' &= 5x+12y-17 \\ y' &= 12x-5y+2 \end{aligned}$$

VI

1. Show that the equation of the circle cutting orthogonally the three circles $x^2+y^2+2g_ix+2f_iy+c_i = 0$, for $i = 1, 2, 3$, is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ c_1 & -g_1 & -f_1 & 1 \\ c_2 & -g_2 & -f_2 & 1 \\ c_3 & -g_3 & -f_3 & 1 \end{vmatrix} = 0, \text{ and reduce this equation to the form}$$

$$\begin{vmatrix} x+g_1 & y+f_1 & g_1x+f_1y+c_1 \\ x+g_2 & y+f_2 & g_2x+f_2y+c_2 \\ x+g_3 & y+f_3 & g_3x+f_3y+c_3 \end{vmatrix} = 0$$

Give the geometric interpretation of the last result.

2. Discuss the conditions that n circles may have a common orthogonal circle.

3. A, B, C, D, E, F are the six vertices of a complete quadrilateral, AC, BD, EF are the diagonals and LMN is the diagonal triangle. If S, S', S'' are circles described on the segments AC, BD, EF as diameters, prove that the circumcircle of the triangle LMN is orthogonal to S, S', S''.

4. If a circle C' when inverted with respect to a circle C becomes a circle C'', prove that C, C', C'' are coaxal.

5. PL, QM are perpendiculars from the points P, Q to the polars of Q and P respectively with respect to a circle having O as the centre. Prove that

$$\overline{OP}.\overline{QM} = \overline{OQ}.\overline{PL}$$

6. Let three circles which pass through a point intersect in pairs in three points forming a curvilinear triangle. Prove by inversion that sum of the angles of the curvilinear triangle is equal to two right angles.

7. Three circles $\Sigma_i\,(i = 1, 2, 3)$ cut a fourth circle Σ orthogonally and intersect in pairs forming a curvilinear triangle within Σ. Prove by inversion that the sum of the angles of the curvilinear triangle is less than two right angles.

8. If

$$\begin{aligned} S &= x^2+y^2+2g_1x+2f_1y+c_1 = 0 \\ S' &= x^2+y^2+2g_2x+2f_2y+c_2 = 0 \end{aligned}$$

are two circles of a coaxal system, show that the limiting points of the system are given by

$$S^2(f_2^2+g_2^2-c_2)-SS'(2g_1g_2+2f_1f_2-c_1-c_2)+S'^2(f_1^2+g_1^2-c_1)$$

9. Two equal circles Σ_1, Σ_2 and a third circle Σ inside Σ_1 of half the radius all touch a straight line in the same point. Show that inversion in Σ_1 followed by inversion in Σ_2 is equivalent to inversion in Σ followed by a reflexion in the common tangent.

10. Show that the operations of inversion with respect to two circles in succession are commutative if the circles are orthogonal.

11. Prove that the circles whose equations are $S = 0$ and $S' = 0$ are inverses with regard to either of the circles $(1/r)S \pm (1/r')S' = 0$, where r and r' are the radii of the circles S, S' respectively.

12. Generalise the following theorem by reciprocation: A tangent to a circle is perpendicular to the radius at the point of contact.

VII

1. Find the equations of the affinity transforming the points

(i) $(0, 0)$, $(1, 1)$, $(1, -1)$ into the points $(2, 3)$, $(2, 5)$, $(3, -7)$,

(ii) $(2, 5)$, $(4, 7)$, $(3, 4)$ into the points $(5, -1)$, $(3, 8)$, $(4, 7)$ respectively.

Point out the reasons in case an affinity which is supposed to transform a given triad of points into another does not exist.

2. Show that an affine transformation transforms the differential equation $d^2y/dx^2 = 0$ into $d^2y'/dx'^2 = 0$.

3. Discuss how the following are transformed by an affine transformation: (i) a segment and its mid-point, (ii) an angle and its bisector, (iii) a regular hexagon, (iv) two circles: C of radius 2 osculated from inside by C' of radius 1, (v) a triangle and its nine-point circle, (vi) a square and its area.

4. Show that an affinity multiplies by the same constant factor k the lengths of all line segments with the same direction. Find the value of k^2 corresponding to the direction θ when the circle $x = \cos\theta$, $y = \sin\theta$ is transformed by the affinity

$$x' = ax+by+c$$
$$y' = a'x+b'y+c'$$

Discuss the nature of the transformation when $dk^2/d\theta = 0$.

5. An affine transformation is supposed to transform a certain figure S into itself. What is known about the transformation if S is (i) a triangle, (ii) a circle, (iii) an ellipse, (iv) a parabola?

How are the above results to be modified if S is not transformed into itself, but into a figure S' which is similar to S?

6. Prove that the only affine transformations which preserve directed angle are the transformations of similarity.

7. Using one of the following properties of a pair of conjugate diameters as definition, derive the others: (i) a diameter bisects all chords parallel to its conjugate, (ii) conjugate diameters are separated harmonically by the two asymptotes of the conic, (iii) the point at infinity on a given diameter is the pole of its conjugate.

8. To every parallelogram inscribed in a given ellipse, there exists a second parallelogram inscribed in the same ellipse whose diagonals are parallel to the sides of the first parallelogram. Show that the area of the second parallelogram is uniquely determined by the ellipse.

9. Show that all parallelograms inscribed in a given ellipse whose diagonals are conjugate diameters have the same area.

10. Determine the affine transformations carrying

(i) the parabola $x^2-2xy+y^2+3x+2y+2 = 0$ into $y^2 = 2x$;

(ii) the ellipse $2x^2-2xy+5y^2-2x-8y+4 = 0$ into $x^2+y^2 = 1$.

VIII

1. Find the equations of the involutions transforming the collinear points

(i) 2 and 7 into -3 and 6 respectively, (ii) 3 and 5 into 1 and 7 respectively. Which one of these is hyperbolic? Find its double points.

2. A, B, C, D are four distinct points of a line. The double points of the involution determined by the pairs A, B and C, D are E, F and the double points of the involution determined by A, C and B, D are $E'F'$. Show that $(EF, E'F')$ is harmonic.

3. Prove that the transformation

$$x' = \frac{a_1x+a_2}{b_1x+b_2}$$

is an involution if $a_1+b_2 = 0$. How can then the equation be reduced to the normal form $x'x = c$?

IX

1. Find the homogeneous coordinates of

(i) the origin, (ii) the point at infinity in the direction of the line $ax+by+c = 0$, (iii) the point at infinity in the direction of the slope 5/6, (iv) the point of intersection of the lines $2x_1-3x_2+4x_3 = 0$ and $x_1+x_2+x_3 = 0$, (v) the line joining the points $(4, 1\ -2)$ and $(1, 1, 0)$.

2. (a) Test the linear dependence and independence of the following points:

(i) $(1, 5, 1)$, $(2, 7, 1)$, $(3, 9, 1)$, (ii) $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$.

(b) Test the linear dependence and independence of the following lines:

(i) $x_1+2x_2+x_3 = 0$, $3x_1+4x_2+2x_3 = 0$, $x_1-7x_2+4x_3 = 0$,

(ii) $x_2-x_3 = 0$, $x_3-x_1 = 0$. $x_1-x_2 = 0$.

3. Show that the coordinates (a_i), (b_i), (c_i), (d_i) of 4 given points, no three of which are collinear, can be so chosen that

$$a_i+b_i+c_i+d_i = 0, \qquad i = 1, 2, 3$$

Deduce that the diagonal points of a complete quadrangle are never collinear. Dualise the above.

4. Find the equation of the pair of tangents to the non-degenerate conic

$$\Sigma\, a_{ij}x_ix_j = 0$$

which pass through the point (r_i). Dualise your answer.

5. Let $a_1, a_2, \ldots, a_n$ be n concurrent lines and $Q_1, Q_2, \ldots, Q_{n-1}$ be $n-1$ fixed points not situated on these lines. If P_1 is a variable point of a_1 and the line P_1Q_1 meets a_2 in P_2, P_2Q_2 meets a_3 in $P_3, \ldots, P_{n-1}\ Q_{n-1}$ meets a_n in P_n, show that the correspondence P_1 to P_n is a perspectivity.

6. Let g and g' be two nonparallel graduated lines marked respectively in inches and centimetres. Show that the lines joining graduations bearing the same number in the two scales are either all parallel or touch a parabola of which g and g' are tangents. Find the points of contact of the parabola with the lines g and g'.

X

1. Find the fixed points and the fixed lines of the collineation

$$\rho x_1' = -2x_2+2x_3, \quad \rho x_2' = -3x_1+x_2+3x_3, \quad \rho x_3' = -x_1+x_2+3x_3.$$

2. In a collineation the vertices of a plane quadrangle $ABCD$ correspond to themselves in the order $BCDA$. Prove that one of the diagonal points is self-corresponding point and the opposite side is a self-corresponding line.

3. In a projective plane three straight lines g, g', g'' are given which are (*a*) concurrent, (*b*) nonconcurrent. For both the cases, discuss the linear transformation T of the plane for which g, g', g'' are invariant. The discussion should refer to the following questions: If P and P' are two different points of the plane not situated on any of the three lines, but arbitrary otherwise, (i) is there any transformation T transforming P into P', and if so, is T uniquely determined? (ii) is T uniquely determined by the additional condition that P remains fixed? (iii) is T uniquely determined by the additional condition that P and P' remain fixed?

4. Show that the correlation

$$\rho u_1 = 2x_1 - x_2 + x_3, \qquad \rho u_2 = -x_1 + x_2 + 3x_3, \qquad \rho u_3 = x_1 + 3x_2 - x_3$$

when carried out twice results in the identity.

5. Show that the triangle with vertices (0, 1, 1), (1, 0, 1), (1, 1, 0) is self-polar with respect to the conic $x_1^2 + x_2^2 + x_3^2 + 2x_2x_3 + 2x_3x_1 - 6x_1x_2 = 0$.

6. Determine a triangle which is self-polar with respect to the conic

$$2u_1^2 - u_2^2 + u_3^2 - 2u_1u_3 - 4u_2u_3 = 0$$

and has the line (1, 2, 1) as one of its sides.

7. If a collineation possesses four fixed points, no three of which are collinear, show that the collineation is the identity.

XI

1. Two projective coordinate systems have the same triangle of reference but different unit points. Determine the form of the transformation from one system to the other.

2. Find the equation of the most general transformation of projective coordinates which introduces the lines

$$2x_1 - 3x_2 + x_3 = 0, \qquad x_1 + 2x_2 - 3x_3 = 0, \qquad x_1 - x_2 + x_3 = 0$$

as the sides $x'_1 = 0$, $x_2' = 0$, $x_3' = 0$ of the new triangle of reference. Then determine the particular transformation which also introduces (2, 3, 2) as the new unit point.

3. Let P_1, P_2, P_3, P_4, P_5, P_6 be six points in a projective plane, no triplet of them being collinear. Denote the point of intersection of P_iP_j, and P_rP_s by $Q(ij\,;\,rs)$, where i, j, r, s run over 1, 2, 3, 4, 5, 6. Prove that if $Q(12\,;\,45)$, $Q(23\,;\,56)$ and $Q(34\,;\,61)$ are collinear, then $Q(12\,;\,45)$, $Q(26\,;\,43)$ and $Q(65\,;\,31)$ are also collinear. Generalise this statement.

4. Show that if three fixed tangents to a parabola cut an arbitrary fourth tangent in the points A, B, C then $\overline{AB}/\overline{BC}$ is constant. Generalise by collineation.

5. The polars of two points P, Q with respect to a conic S meet S in the point pairs A, B and C, D respectively. Prove that the conic S' which touches QA at A and passes through B, C, D also touches QB, PC, PD at B, C, D respectively.

6. Given that the locus of a point such that the tangents drawn from it to two given conics S and S' form a harmonic pencil is a conic F which passes through the points of contact of S and S' with their common tangents. Deduce that the condition that S and S' may have double contact is that the matrix of the coefficients of S, S' and F is of rank 2.

7. A and B are two fixed points and AP and BP are conjugate lines with respect to a conic S. Show that the locus of P is a conic S' and find the points on S where it is cut by S'.

8. If three conics circumscribe the same quadrangle, prove that the points of contact of common tangent to any two is separated harmonically by its points of intersection with the third conic.

9. Given five tangents to a conic, show how to determine (i) the point of contact on any one of them, (ii) the other tangent to the conic from a point on any one of the given tangents.

10. Show how to construct the tangent at one of five given points through which a non-degenerate conic is to pass.

11. A triangle is inscribed in a conic and two of its sides pass through two fixed points. Find the envelope of the third side and show that it is degenerate when the two fixed points are conjugate with respect to the conic.

12. Describe how to construct the conic with respect to which two given perspective triangles are reciprocal polar triangles.

13. Find the triangles of reference with respect to which the equations of conics can be reduced to the following normal forms :

$$x_1^2+x_2^2-x_3^2 = 0, \quad x_1^2-x_2^2-x_3^2 = 0, \quad x_1^2-2x_2x_3 = 0$$

XII

1. If a, b, c are the direction-cosines of a directed line g, prove that the orthogonal projection of a vector (a', b', c') on g is $aa'+bb'+cc'$.

2. If d is the length of a vector and d_1, d_2, d_3 are the lengths of its orthogonal projections on the coordinate planes, prove that $2d^2 = d_1^2+d_2^2+d_3^2$.

3. Prove that the lines

$$(x-a)/a' = (y-b)/b' = (z-c)/c' \quad \text{and} \quad (x-a')/a = (y-b')/b = (z-c')/c$$

intersect, and find the coordinates of the point of intersection and the equation of the plane in which they lie.

4. Find the length and the equation of the common perpendicular to the lines

$$x = -(y-11)/2 = z-4 \quad \text{and} \quad (x-6)/7 = -(y+7)/6 = z.$$

5. Show that the shortest distance between two opposite edges a, d of a tetrahedron is $6V/ad \sin\theta$, where θ is the angle between the edges and V is the volume of the tetrahedron.

6. Find the spatial arrangement of the four planes whose equations are given by

$$\begin{array}{rl@{\qquad}rl}
\text{(i)} & x-2y+3z = 7 & \text{(ii)} & 2x+3y+4z = 1 \\
& 2x+y-z = 5 & & 5x+8y+13z = 1 \\
& 3x-y+2z = 12 & & x+y-z = 2 \\
& x+8y-11z = -11 & & 3x+5y+9z = 1
\end{array}$$

7. Ler ϵ and ϵ' be two planes intersecting in a line s, and let the triangle $A'B'C'$ in ϵ' be the projection of the triangle ABC in ϵ from a centre of projection V. Rotate ϵ about s. Prove that (i) $A'B'C'$ will continue to be the projection of ABC and (ii) the locus of V is a circle lying in a plane perpendicular to s.

8. $A_1\ B_1\ C_1$ and $A_2\ B_2\ C_2$ are two straight lines and $B_1\ B_2$ is the shortest distance between them. If C_1, C_2 are any two points on the lines such that $C_1\ A_2$ is perpendicular to $A_2\ B_2\ C_2$ and $C_2\ A_1$ is perpendicular to $A_1\ B_1\ C_1$, prove that

$$\overline{A_1 B_1}\cdot\overline{B_1 C_1} = \overline{A_2 B_2}\cdot\overline{B_2 C_2}$$

XIII

1. Two projective planes of points are said to be perspective if the lines joining the corresponding points are concurrent. Show that a necessary and sufficient condition that two projective planes of points be perspective is that each point common to the two planes be self-corresponding.

State and prove the dual of the above.

2. Two projective pencils of lines lie in two different planes but have a common centre. Discuss the envelope of the planes passing through the corresponding lines of the two projective pencils.

3. (a) Find the Pluecker coordinates of the edges of the tetrahedron of reference.

(b) Describe the lines whose Pluecker coordinates satisfy the equations :

(i) $p_{23} = 0$ (ii) $p_{23} = 0 = p_{14}$ (iii) $p_{23} = 0 = p_{31}$

XIV

1. If A is an arbitrary affine transformation and T is an arbitrary translation, show that

ATA^{-1} is a translation.

[On account of this property, the subgroup of translations which is thus transformed into itself is called a *normal subgroup* of the affine group.]

2. In the extended Cartesian space of n dimensions, show that the affine transformations are the collineations which leave the hyperplane at infinity ($x_{n+1} = 0$) fixed and the similarities are the affine transformations which leave the *absolute* ($x_1^2 + \ldots\ldots + x_{n+1}^2 = 0,\ x_{n+1} = 0$) fixed. Further, in the non-extended space, show that the orthogonal transformations are the similarities for which the distance ($\sqrt{\{(x_1-y_1)^2 + \ldots\ldots + (x_n-y_n)^2\}}$) remains invariant.

3. Classify the following relations as affine or metric, giving reasons :

(i) a triangle and its orthocentre, (ii) a conic and its eccentricity,

(iii) a central conic and a pair of its conjugate diameters,

(iv) ratio of distances on the same or two parallel straight lines.

4. Are the following classes of curves affine and, in particular, similar :

(i) all ellipses, (ii) all parabolas, (iii) all hyperbolas, (iv) all circles ?

XV

1. Show that the surface $x_1^2/a^2 - x_2^2/b^2 - 2cx_3x_4 = 0$, $abc \neq 0$, can be transformed by a collineation into $x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0$.

2. Characterise the quadric whose equation is $y^2 + z^2 + 2(y+z)(x-w) = 0$, where x, y, z, w are homogeneous coordinates. Consider specially the case when $z = 0$ is the plane at infinity.

3. Obtain the equation of the locus of common transversals of the lines

$$x_2 = x_3 = 0,\quad x_1 = x_2 + x_4 = 0,\quad x_1 + x_4 = x_2 + x_4 = 0$$

4. Show that given four projective rows $[P]$, $[Q]$, $[R]$, $[S]$ on four arbitrarily chosen lines, there are, in general, 4 and only 4 planes which contain corresponding points P, Q, R, S.

XVI

1. Q_1 and Q_2 are two non-degenerate quadrics. Let ε run over all planes and let P_1, P_2 be the poles of ε with respect to Q_1, Q_2 respectively. Show that there exists a unique collineation which carries every point P_1 into the corresponding point P_2 and another carrying P_2 and P_1.

Discuss the conditions under which the above two collineations become identical, and show that this is the case when Q_1 and Q_2 are given by

$$x_1^2+x_2^2+x_3^2+x_4^2 = 0 \qquad \text{and} \qquad x_1x_2-x_3x_4 = 0$$

Show that the collineation has two lines of fixed points and two pencils of fixed planes.

2. Let Q be a nondegenerate quadric, and let T be a tetrahedron which is self-polar with respect to Q. The polar field of Q generates on each of the edges of T an involution of the points of the edge as well as an involution of the planes passing through the edge. Discuss the types (elliptic or hyperbolic) of these twelve involutions according to the projective character of Q.

3. Show that the lines of a pencil of lines whose centre and plane do not belong to a quadric are conjugate in pairs with respect to the quadric, and that these pairs form an involution.

XVII

1. Is it possible to transform a sphere into itself by a collineation which is not an affinity?

2. Prove that there is a unique diametral plane of a central quadric which is conjugate to a given diameter, and that the line at infinity in the plane and the point at infinity of the diameter are polar and pole with respect to the conic at infinity on the quadric. Discuss conjugate diameteral planes and conjugate diameters.

XVIII

1. Prove that the rigid motion

$$x' = x\cos\theta - y\sin\theta, \qquad y' = x\sin\theta + y\cos\theta, \qquad z' = z+c$$

is a *screw motion* about the z-axis, that is, the product of a rotation about the z-axis and a translation parallel to the z-axis.

2. Show that every affine transformation transforms
   (i) a ruled surface into a ruled surface.
   (ii) a paraboloid into a paraboloid.

3. Let A be an affinity which keeps an ellipsoid E invariant. Prove that there exists a diameter of E such that every point of the diameter is invariant for A.

4. Show that the affinity

$$x' = ax, \qquad y' = by, \qquad z' = cz$$

carries three mutually perpendicular (and therefore conjugate) diameters of the sphere $x^2+y^2+z^2 = 1$ into three conjugate diameters of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2 = 1$. Hence show that the sum of the squares of the lengths of three conjugate diameters of an ellipsoid is constant.

XIX

1. Apply a rigid motion to reduce the equation

$$2x^2-y^2-z^2-2yz-4x+6y+2z+2 = 0$$

to its normal form. What surface does it represent ?

2. Verify that the surface given by

$$x = a\cos(\theta-\phi)/\cos(\theta+\phi),\quad y = b\cos\theta\sin\phi/\cos(\theta+\phi),\quad z = c\sin\theta\cos\phi/\cos(\theta+\phi)$$

is a hyperboloid of one sheet, whose generating lines are $\theta =$ const. and $\phi =$ const. Find the Cartesian equation of the surface.

3. Prove that the locus of intersections of perpendicular planes through two skew lines g_1 and g_2 is a hyperboloid whose real circular sections are given by planes perpendicular to g_1 and g_2.

4. Tangents to a hyperboloid of one sheet H passing through a point P form a cone which osculates H along a conic $C(P)$. Discuss the position of P when $C(P)$ is known to be a hyperbola, a parabola, an ellipse, a circle.

5. Prove that a cone possesses either no set of three mutually orthogonal generators, or an unlimited number.

6. Show that the pencil of planes through a generator of one system of hyperboloid of one sheet cuts the surface in the generators of the other system, and that these planes are tangent planes to the surface at the points where they meet those generators.

7. Show that the cross-ratio of four points on a generator of a hyperboloid of one sheet is equal to the cross-ratio of the four tangent planes at the four points.

8. Show that the section of the surface

$$xy+yz+zx = c^2$$

by the plane $\lambda x+\mu y+\nu z = \rho$ will be a parabola if $\sqrt{\lambda}+\sqrt{\mu}+\sqrt{\nu} = 0$; and that of the surface

$$x^2+y^2+z^2-2yz-2zx-2xy = c^2$$

will be a parabola if $\lambda\mu+\mu\nu+\lambda\nu = 0$.

9. Prove that the perpendicular distance from the centre to the tangent plane at an umbilic of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2 = 1$ is ac/b.

10. Prove that a straight line which always intersects two given lines and is perpendicular to one of them generates a hyperbolic paraboloid.

11. Two plane mirrors are inclined at a fixed angle and a ray of light is reflected between them. Show that all the reflected rays will lie on a hyperboloid of revolution.

12. Let an eye be placed on the surface of a hyperboloid whose equation is $ax^2+by^2+cz^2 = 1$. Prove that the points, the generating lines through which appear to be perpendicular, will lie in a plane whose equation is

$$2(a^2fx+b^2gy+c^2hz) = (a+b+c)\,(afx+bgy+chz-1),$$

where (f, g, h) is the position of the eye.

13. Show that the equations

$$x^2/(a^2-\lambda)+y^2/(b^2-\lambda) = 2z-\lambda, \quad a^2 > b^2,$$

where λ is a parameter not taking the values a^2 or b^2, represent three families of confocal paraboloids defined for the following values of the parameter

$$b^2 > \lambda, \quad a^2 > \lambda > b^2, \quad \lambda > a^2$$

and show that they form a triply orthogonal system.

14. Prove that every quadric which does not touch the plane at infinity is either a surface of revolution or has two systems of circular sections.

XX

1. Transform the following homogeneous quadratic forms

(i) $4x_1^2+9x_2^2-16x_3^2-8x_4x_5$

(ii) $x_1^2-x_2^2-x_3^2-2\,x_2x_3$

to their normal forms by nonsingular linear transformations.

2. Verify that the numbers of terms in the normal forms obtained above are equal to the ranks of the matrices of the coefficients of the given quadratic forms. What are the signatures of these forms ?

INDEX

[*Numbers refer to pages*]

Affinity, Affine transformations, 70, 168, 218, 220, 257, 258
Affine, geometry, 168, 221
—property, 221, 249
Angle, 6, 188
Asymptotes, 38, 174
Asymptotic cone, 273
Axis, of coordinate, 3, 179
—of homology, 212
—of a pencil of planes, 201
—of projection, 193
—of similitude, 62
radical—, 56
Bundle, 199
Brianchon's theorem, 150
Coordinates,
areal—, 177
Cartesian—, 4, 180
homogeneous—, 104, 108
Pluecker—, 203
projective—, 131, 132
trilinear—, 175
Centre, of a conic, 30
—of involution, 92, 103
—of a pencil of lines, 16
—of projection, 193
—of a quadric, 249
—of similitude, 60
Ceva's theorem, 124
Circle, general, 54—69
—at infinity, 245, 246, 249, 250
—of similitude, 61
coaxal—, 56, 57
nine-point—, 177
Circular, points at infinity, 173
—sections, 279
Classification, of conics
affine—, 77
metric—, 33
projective—, 139
—of geometries, 221
—of quadrics,
affine—, 273
metric—, 265
projective—, 229, 231
Collineations, 115, 134, 226
Complete, quadrangle, 101
—quadrilateral, 101
Complex, plane, 42
—of lines, 204
Cone, of the second degree, 234
tangent—, 263
Conic,
apolar—, 166, 167
—at infinity, 250
confocal—s, 164
degenerate and non-degenerate—, 29
harmonic—, 167
pencil of—s, 156
nine-point—, 158
optical properties of—, 39, 40
line—, 148
point—, 148
projective generation of—, 146
Conjugate, diameters, 79—82
—lines 37, 226
—planes 225
—points, 37, 225
Correlations, 127, 236
Cross-ratio, 15—23, 206, 208
Cylinder, parabolic, elliptic, hyperbolic 266, 268, 270
Desargues' theorems, 124, 155, 212
Diagonal points, diagonal triangle, 101
Dilation, 51
Dimensionality of spaces, 214
Directed segments, 2, 180
Discriminant of a conic, 29
Duality, 111
Eccentric angle, 84, 85
Eccentricity of a conic, 38
Ellipse, 31
Ellipsoid, 250

Envelope, 113, 287
Equivalence, 222
Foci of conics, 38, 44, 45
Geometry,
  affine—, 221
  metric—, 221
  projective—, 221
Groups, of transformations, 216
  subgroups of—, 216
Harmonic division, 22
Hessian normal form 11, 186
Homology, 211
Homothetic transformation, 52, 219, 220
Hyperbola, 31, 41
Hyperbolic paraboloid 268—270
Hyperplane, 215
  —at infinity, 218
Inertia, law of—, 292
Invariants of transformation, 25, 221
Inversion, 62
Involution, 86
  affine—, 121
  double points of—, 92
  elliptic—, 89
  hyperbolic—, 89
Isotropic lines, 43
  —tangents, 44, 45
Klein, 221
Laguerre, 45
Line at infinity, 96
Line complex 204
Linear dependence,
  —of lines, 105
  —of points, 105, 227
Linear transformation, 48
Menelaus' theorem, 124
Metric geometry, 221
  —property, 221
Möbius sheet, 136
Normal forms, 33, 77, 138, 171, 265
Nucleus of polarity, 129, 238
Null system, 237, 238
Oriented plane, 188
Paddle motion, 49
Pappus' theorem, 125
Parabola, 33
Paraboloids, 268—270, 281, 282
Pascal's theorem 149
Pencil, of conics, 156
  —of lines, 86
  —of planes, 199
Perspectivity, 87, 206
Plane, at infinity, 202, 247, 248, 250
  —fields, 119, 231
  —sections, 277
Polar, 36, 56, 128, 225, 262
  —fields, 129, 242, 243
  lines, 226
  —planes, 225, 263
  —reciprocation, 65
  —triangles, 128, 225
  —tetrahedron, 225
Polarity, 127, 128, 236—238
Pole, 36, 56, 128, 225, 262
Power of a point, 55
Projective, coordinates, 131—132
  —geometry, 221
  —property, 221
  —transformations, 116
Projectivity, 86, 207
Quadratic, dependence of points, 141
  —form, 230, 264, 265, 290
Quadrics, 224
  hyper—, 294
  confocal—, 284
  singular—, 224
Range, 15, 197
Reflexion, 51
Regulus, 233
Rigid motion, 24, 220, 252
Rotations, 26, 220, 254
Row of points, 15, 197
Ruled surface, 287
Self-conjugate, tetrahedrons, 225
  —triangles, 56, 128, 225
Sheaf, 199
Shear, 74
Signature, 293
Similarities, 51—52, 218, 220
Skew lines, 183
Space, 179, 214
Standt's theorem, 152
Strain, 74
Surfaces of revolution, 286
Sylvester, 292
Symmetries, 47, 48, 252

Tangent, line, 224
    —plane, 225
Tangential equation, 239
Tetrahedron of reference, 226
Triangle of reference, 133
Transformations,
    associative law for—, 216
    bilinear—, 88
    conformal—, 63
    homographic—, 88, 93
    inverse of—, 216
    one-dimensional—, 120
    orthogonal—, 251
    product of—, 216
    two-dimensional—, 120
Translations, 26, 220
Umbilic, 284
Unit point and unit line, 133
Vanishing point and line, 193
Vectors, 2, 4, 180
Vertex of a conic, 38, 39

23-12-60